

A foundation for probabilistic beliefs with or without atoms

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A FOUNDATION FOR PROBABILISTIC BELIEFS WITH OR WITHOUT ATOMS

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Abstract

We provide sufficient conditions for a qualitative probability (Bernstein, 1917; de Finetti, 1937; Koopman, 1940; Savage, 1954) that satisfies *monotone continuity* (Villegas, 1964; Arrow, 1970) to have a unique countably additive measure representation, generalizing Villegas (1964) to allow atoms. Unlike previous contributions, we do so without a cancellation or solvability axiom.

First, we establish that when atoms contain singleton cores, *unlikely cores*—the requirement that the union of all cores is not more likely than its complement—is sufficient (Theorem 3). Second, we establish that *strict third-order atom-swarmling*—the requirement that for each atom A , the less-likely non-null events are (in an ordinal sense) more than three times as likely as A —is also sufficient (Theorem 5). This latter result applies to intertemporal preferences over streams of indivisible objects.

Keywords: beliefs, qualitative probability, atoms, atom-swarmling.

1 Introduction

1.1 Executive summary

From the doctor's choice of treatment, to the employer's choice of job applicant, to the investor's choice of portfolio, to the mortal's choice of religion, and beyond: much behavior, including much of the economic behavior we observe and strive to model, is the selection of an action with uncertain consequences. Our standard model is founded on the postulate that such choices, when made by someone who is rational, can be decomposed into (1) beliefs about the relative likelihood of events, and (2) tastes among outcomes (Ramsey, 1931). This article revisits a classic question: when are such beliefs consistent with standard probability theory?¹

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¹Machina and Schmeidler (1992) call this question the first of two lines of inquiry culminating in the modern theory of subjective probability.

More precisely, suppose we are given a nonempty set of states S , a σ -algebra of events $\mathcal{A} \subseteq 2^S$ with $S \in \mathcal{A}$, and a *qualitative probability* \succsim on \mathcal{A} : a binary relation on \mathcal{A} , consisting of comparisons of events on the basis of relative likelihood, satisfying minimal probabilistic requirements (Bernstein, 1917; de Finetti, 1937; Koopman, 1940; Savage, 1954). When does \succsim admit representation by a σ -measure² $\mu : \mathcal{A} \rightarrow [0, 1]$?

A necessary condition is *monotone continuity* (Villegas, 1964; Arrow, 1970): if $B_1 \supseteq B_2 \supseteq \dots$, and for each $i \in \mathbb{N}$, $B_i \succsim A$, then $\bigcap B_i \succsim A$.³ On the appeal of this axiom, Arrow writes: “The assumption of Monotone Continuity seems, I believe correctly, to be the harmless simplification almost inevitable in the formalization of any real-life problem.” While this continuity axiom is not sufficient on its own (Kraft, Pratt, and Seidenberg, 1959), we obtain sufficient conditions when it is paired with the requirement that there are no *atoms*: non-null events for which each subset is either equally-likely or null (Villegas, 1964).

Though the practice of using σ -measures to represent beliefs with atoms is prevalent throughout economic analysis (particularly when the state space is discrete), atoms present non-trivial technical challenges for our problem. In fact, the large literature dedicated to our problem can be classified according to which of the following is imposed:

- there are no atoms;
- the qualitative probability satisfies a “cancellation” axiom;
- the qualitative probability satisfies a “solvability” axiom; or
- there are additional primitives beyond S , \mathcal{A} , and \succsim .

The contribution of this article is to proceed without imposing any of the above.

First, we show that under *monotone continuity*, if each atom contains a *core*—a singleton atom—and if moreover the union of all cores is at most as likely as its complement, then unique σ -measure representation is guaranteed (Theorem 3). In this case, there is an infinitely-divisible event whose probability is at least one-half.

What if each infinitely-divisible event is less likely than its complement? In particular, what if there are no infinitely-divisible events at all? To address this case, we propose *strict third-order atom-swallowing* (3^+ -AS): if there are any atoms at all, then each is sufficiently overwhelmed by less-likely non-null events.⁴ Formally, for each atom A , there is a pairwise-disjoint collection of less-likely non-null events that can be partitioned into three⁵ subcollections, with the union of each more likely than A (Figure 1). Our main result is that *monotone continuity* and 3^+ -AS are sufficient to guarantee σ -measure representation.

²In this article, a *measure* is a finitely-additive probability measure, σ -*additivity* is *countable-additivity*, and a σ -*measure* is a σ -additive measure.

³Villegas (1964) and Arrow (1970) give different statements that are logically equivalent; I owe this particularly elegant statement to a referee.

⁴Earlier versions of this article further required that these non-null events were in fact atoms, ruling out the possibility of a finite collection of cores. I am indebted to a referee for suggesting this generalization.

⁵Why three? Why not the analogous 1^+ -AS or 2^+ -AS? This is addressed over the course of our proof’s explanation. For now, we remark only that *at least three* has appeared in earlier research on a similar topic: given an ordering over a product space, if there are only two “essential” factors, an additional condition is used to guarantee an additive representation (Debreu, 1959). For rough intuition: while addition is a binary operation, its cancellation property is articulated with three elements.

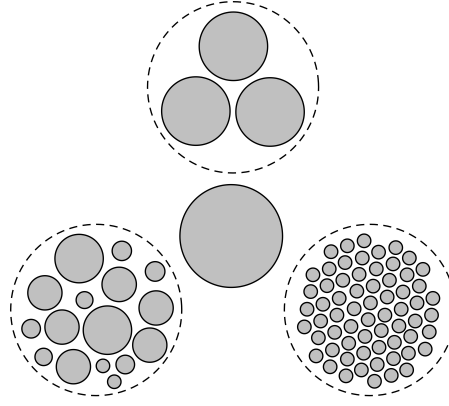


Figure 1: Strict third-order atom-swarming. In the illustration, each gray circle is a non-null event, and a more likely event has a greater area. The center event is an atom, and the smaller events are pairwise-disjoint. Each dashed circle shows a collection of these smaller events whose union is more likely than the center atom; in this way, the atom is ‘sufficiently swarmed’ by the smaller events. The top collection includes a few relatively large events, the rightmost collection includes many relatively small events, and the leftmost collection includes events of different sizes. Not pictured, but allowed, is an infinite collection of events. The axiom requires that there are three such collections for *each* atom.

1.2 Related literature

According to the historian of probability Oscar Sheynin ([Sheynin, 2005](#)), Boole was the first to call for an axiomatic foundation for probability,⁶ and this call was prominently reiterated over four decades later in Hilbert’s Sixth Problem.⁷ It would be another three decades before Kolmogorov provided today’s standard axioms for cardinal probability, explaining that this task was “a rather hopeless one before the introduction of Lebesgue’s theories of measure and integration” ([Kolmogoroff, 1933](#)). Kolmogorov’s axioms involve σ -algebras, which are appropriately a subclass of the logical algebras studied by Boole: anticipating Kolmogorov, Boole wrote that “the theory of probabilities is coextensive with that of logic” because “as simple events are expressed by simple propositions, so combinations of events are expressed by compound propositions” ([Boole, 1851](#)).⁸

⁶From [Boole \(1854\)](#): “It cannot, I think, be doubtful that the theory of probabilities belongs to that class of sciences which are termed pure sciences[...]. From this idea, from the definition of the *measure* of probability by which it becomes associated with a number, and from the laws of thought with which it is connected through its having to do with events capable of logical expression, flow the axioms and first principles of the science[...]. I go on to observe, that such principles, if truly axiomatic, lead in every pure science, and therefore in the theory of probabilities, to a developed system of truth, or of methods for the attainment of truth, which possess certain *invariable* characteristics never found unimpaired where error has been permitted to enter.”

⁷From [Hilbert \(1900\)](#): “The investigations on the foundations of geometry suggest the problem: to treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.” The investigations Hilbert refers to constitute the earliest axiomatic work in science that survives to this day ([Euclid, circa 300 BC](#)).

⁸Indeed, the modern notion of an abstract Boolean algebra ([Halmos, 1963](#)) involves ‘propositions’ that, by Stone’s Representation Theorem, can be viewed as events of a state space without loss of generality ([Birkhoff, 1935](#); [Stone, 1936](#)). This is not true for σ -algebras; the lost generality is made explicit by the Loomis-Sikorski Representation Theorem ([Loomis, 1947](#); [Sikorski, 1960](#)). Interestingly,

This intimate relationship between logic and probability does not, however, require a cardinal concept. Indeed, while a cardinal concept is natural for a *frequentist* interpretation of probability, in which “a probability is the relative frequency of some kind of event in a certain type of sequence of events or, according to some, in a set of events” (Savage, 1961), this interpretation is rejected by *Bayesian* statisticians, who advocate reaction to evidence using Bayes’ Theorem (Bayes, 1763).⁹

An alternative to frequentist probability, favored by many Bayesians, is a *subjective* (or personal) interpretation of probability (Ramsey, 1931), and perhaps the most natural concept for this interpretation is not cardinal, but ordinal: when confronted with two logical propositions whose truth is not known, one is able to compare the two, as though using a balance, to form a *belief* about their relative likelihood. In fact, Bernstein axiomatically pursued just such a concept even before Kolmogorov (Bernstein, 1917)!¹⁰

This article’s question is one of harmony: when do the concepts of cardinal probabilities and ordinal beliefs coincide? When is it *as though* instead of comparing two propositions with a balance, one instead compares them using a scale; first weighing the first proposition, then weighing the second, and finally comparing the two weights? What assumptions, exactly, are we making when we model someone’s beliefs with a probability measure? We categorize the previous literature dedicated to this question by which of four assertions is imposed, emphasizing that our contribution is to proceed without any of these assertions.

The literature with no atoms

The seminal contributions to the qualitative probability literature (Bernstein, 1917; de Finetti, 1937; Koopman, 1940) imposed that S can be partitioned into an arbitrarily large number of equally likely events. This implies that there is a unique measure that “almost represents” \succsim , though it may assign the same probability to two distinguished events (see Kreps, 1988).

To guarantee representation, Savage (1954) imposed a stronger axiom, *fineness-and-tightness*, which moreover implies that the representation’s range is $[0, 1]$.¹¹ Surprisingly,

while Stone’s result relies on the Boolean Prime Ideal Theorem, which is weaker than the Axiom of Choice (Halpern and Lévy, 1964), the result of Loomis and Sikorski does not (Buskes, de Pagter, and van Rooij, 2008), nor does the fact that no generality is lost for finite algebras (Birkhoff, 1937).

⁹As Savage (1961) explains: “Whatever else a frequentist may do with the results of an investigation he cannot, as a frequentist, use them to calculate probabilities of the uncertain propositions that are under investigation. Technically, this means that he is cut off from most applications of Bayes’ theorem, the algorithm for calculating what the new probability of a proposition is on the basis of its original probability and new relevant evidence. One’s natural inclination to ask, ‘To what degree of conviction does this new data entitle me?’ usually must be, and has been, regarded as a nonsense question by the frequentist. The frequentist is required, therefore, to seek a concept of evidence, and of reaction to evidence, different from that of the primitive, or natural, concept that is tantamount to application of Bayes’ theorem.”

¹⁰At the time, Bernstein’s work was not well-known outside of Russia, and today the independent works of de Finetti and Koopman remain better-known (de Finetti, 1937; Koopman, 1940). Slutsky (1922) is another important Russian contribution toward axiomatic probability from this time; citing it in his obituary of Slutsky, Kolmogorov (1948) remarked that Slutsky “was the first to draw a correct picture of the purely mathematical essence of probability theory.” Slutsky (1925), in turn, remarked: “Regrettably, in 1922 I was not acquainted with Bernstein (1917) where the indicated ideas had already been realized, and more formally at that. That contribution deserves to be most seriously studied.” See Sheynin (2010) for a discussion, and see Sheynin (2005) for a translation of Bernstein (1917).

¹¹For simplicity of presentation, Savage further imposed that $\mathcal{A} = 2^S$, but his analysis is valid if \mathcal{A} is any σ -algebra (Wakker, 1981; Wakker, 1993a). If \mathcal{A} is only required to be an algebra and a weaker

Savage’s axioms are in fact compatible when $|S| = |\mathbb{N}|$; they are only incompatible when S is finite (Kreps, 1988; Wakker, 1993a). That said, Savage’s axioms are incompatible with atoms, and they are incompatible with σ -additivity when $|S| = |\mathbb{N}|$.¹²

Savage deliberately avoided any continuity axiom implying σ -additivity, explaining: “I know of no argument leading to the requirement of countable additivity[. . .]it therefore seems better not to assume countable additivity outright as a postulate, but to recognize it as a special hypothesis yielding, where applicable, a large class of useful theorems.” But there have since been two particularly strong such arguments: first, σ -additivity is required for avoiding money pumps (Adams, 1962; Seidenfeld and Schervish, 1983), and second, σ -additivity is required to ensure that choice always respects strict first order stochastic dominance (Wakker, 1993b). Based on these observations and others, Stinchcombe (1997) argues that a measure which is only finitely-additive indicates a misspecified state space: “One summary[. . .]is that countably infinite constructions require countably additive probabilities.”

Villegas (1964) identified the appropriate continuity axiom: if μ is a measure representation of \succsim , then μ is a σ -additive if and only if \succsim is *monotonely continuous* (Theorem V1). Furthermore, (i) *monotone continuity*, and (ii) the requirement that there are no atoms, are sufficient for σ -measure representation (Theorem V2). We appeal to these important results directly in our proof. Unfortunately, these axioms are incompatible when $|S| \leq |\mathbb{N}|$.

While Kopylov (2010) does not explicitly study qualitative probabilities, it is clear from his analysis that σ -measure representation is guaranteed by imposing only *strong monotone continuity* when \mathcal{A} is *countably separated* (Mackey, 1957). That there are no atoms is then implied. Because Kopylov’s axioms are incompatible when $|S| \leq |\mathbb{N}|$, and because \mathcal{A} can only be countably separated when $|S| \leq |\mathbb{R}|$,¹³ it follows that under the Continuum Hypothesis, this approach is custom-tailored to the case where $|S| = |\mathbb{R}|$.

Our results generalize that of Villegas (1964) by weakening the requirement that there are no atoms. Like Savage (1954), our axioms are incompatible only when S is finite. Unlike the rest of this part of the literature, our results are compatible with atoms, and with σ -additivity whenever S is infinite.

The literature with cancellation

Even when S is finite, there are qualitative probabilities without measure representations (Kraft, Pratt, and Seidenberg, 1959). This was first demonstrated by means of

version of *fineness-and-tightness* is imposed, then a measure representation is still guaranteed (Wakker, 1981), but its range need only be a dense subset of $[0, 1]$ (Marinacci, 1993). While there are indeed such atomless measures without convex range (Nunke and Savage, 1952), every atomless σ -measure has convex range (see Halmos, 1947, Lemma 2).

¹²Savage’s simplifying assumption that $\mathcal{A} = 2^S$ seriously restricts compatibility with σ -additivity. If $|S| = |\mathbb{R}|$, then the existence of an atomless σ -measure defined on 2^S is inconsistent with the Continuum Hypothesis (Banach and Kuratowski, 1929; Ulam, 1930). Savage avoids this issue by relaxing σ -additivity, as there are well-behaved atomless measures—for example when $S = [0, 1]$, there are measures that (1) agree with the Lebesgue measure on those sets where it is defined, and (2) assign the same number to any pair of congruent sets (Banach, 1932). By contrast, we avoid this issue both by allowing \mathcal{A} to be any σ -algebra and by allowing atoms.

¹³ \mathcal{A} is *countably separated* if there is a countable collection of events $\mathcal{A}^* \subseteq \mathcal{A}$ such that for each distinct pair $s, s' \in S$, there is $A \in \mathcal{A}^*$ with $s \in A$ and $s' \notin A$. That this implies $|S| \leq |\mathbb{R}|$ can be seen from the following argument of Faris (2007): index \mathcal{A}^* by the natural numbers, so that $\mathcal{A}^* = \{A_1, A_2, \dots\}$, and define the mapping φ from S to the Cantor set \mathbb{C} (Cantor, 1883) by defining, for each $i \in \mathbb{N}$ and each $s \in S$, $\varphi_i(s) \equiv \mathbb{1}_{A_i}(s)$. Then φ is an injection, so $|S| \leq |\mathbb{C}| = |\mathbb{R}|$.

a qualitative probability \succsim on the subsets of $\{s_1, s_2, s_3, s_4, s_5\}$ that includes the comparisons:

- (1) $\{s_1\} \succsim \{s_2, s_4\}$
- (2) $\{s_3, s_4\} \succsim \{s_1, s_2\}$
- (3) $\{s_2, s_5\} \succsim \{s_1, s_4\}$
- (4) $\{s_1, s_2, s_4\} \succ \{s_3, s_5\}$

A necessary and sufficient condition for measure representation is *finite cancellation* (Kraft, Pratt, and Seidenberg, 1959; Scott, 1964; Fishburn, 1970; Krantz, Luce, Suppes, and Tversky, 1971), the assertion that there is no such list of comparisons. This condition is quite strong, alone necessary and sufficient for additive representation (Fishburn, 1970). Attempts to justify this axiom generally rely on enriched models—for example, preferences over multi-sets of states, where such a multi-set is interpreted as a portfolio of Arrow-Debreu-McKenzie securities—and as such it can be difficult to interpret.

Stronger and more complex conditions have been identified that guarantee measure representation while allowing atoms when S is infinite (Domotor, 1969; Chateauneuf and Jaffray, 1984; Chateauneuf, 1985). Chateauneuf (1985) in fact provides necessary and sufficient conditions using an axiom that implies *finite cancellation*. We recommend Fishburn’s survey for more information (Fishburn, 1986).

By contrast, we do not impose any cancellation axiom. For the purely discrete case, we instead follow Cantor and Debreu (Cantor, 1895; Debreu, 1954; Debreu, 1964) by constructing an order-dense family of equivalence classes. We then define a binary operation on this family and, following Peano (1889), show that this binary operation interacts appropriately with succession—that is, addition by one—from which we derive cancellation, commutativity, associativity, and all other properties of ordinary addition.

The literature with solvability

To our knowledge, there are two articles studying qualitative probabilities with atoms that do not impose cancellation (Abdellaoui and Wakker, 2005; Chew and Sagi, 2006). Both approaches involve *solvability*:¹⁴ for each pair of disjoint events, there is a subevent of one that is as likely as the other. Abdellaoui and Wakker (2005) allow for mosaics instead of σ -algebras and allow for measures that are not convex-ranged, while Chew and Sagi (2006) work with an ordering of events induced from preferences over acts through “exchangeability” and proceed without *monotonicity*. In both cases there may be atoms, provided any pairwise disjoint collection of atoms is finite with equally-likely members.

By contrast, our results are compatible with an ‘exhaustive’ pairwise disjoint collection of atoms which is finite or infinite, and moreover, the atoms need not be equally-likely. For the discrete case, we explain the difference in approach with a visual metaphor where size corresponds to likelihood. Given two events, one larger than the other, we would often like to find a piece of the larger that is the same size as the smaller. In the *solvability* approach, the larger event is made of a fabric from which the desired piece can be cleanly cut. In our approach, we cannot always create the desired piece because the larger event

¹⁴This is not quite the language used in either article. Abdellaoui and Wakker (2005) use “solvability” to refer to a stronger axiom, while Chew and Sagi (2006) use “completeness” to refer to the given axiom and use “solvability” to refer to a related property for measures.

may consist of blocks that cannot be cut. Nevertheless, whenever the larger event has a rich enough composition, the desired piece can be mosaicked by iteratively adding smaller and smaller blocks to better-approximate the specified size.

The literature with additional primitives

We refer to works that use the entire [Savage \(1954\)](#) model: a state space S , an outcome space X , and preferences over *acts* $f : S \rightarrow X$. Often implicitly, $\mathcal{A} = 2^S$. To uncover the embedded qualitative probability, first select a pair of outcomes x^* and x such that the act guaranteeing x^* is preferred to that guaranteeing x . Beliefs can then be defined as preferences among simple bets: $A \succsim B$ if and only if the act returning x^* on A and x otherwise is at least as desirable as the act returning x^* on B and x otherwise. It is imposed that this is well-defined; \succsim does depend on the choice of x^* and x .

Savage used only \succsim to deliver the unique measure representation of beliefs. But to handle atoms that are not equally likely without using a cancellation or solvability axiom, others have used the entire model, particularly when S is discrete. Typically this involves imposing, at a minimum, that X has cardinality of at least the continuum and that X has a rich topological structure. For example, X might be a simplex of objective “roulette lotteries” that can be mixed (as in [Anscombe and Aumann, 1963](#) and the vast literature that followed), or an interval of dollar amounts (see [Wakker, 1989](#); [Gul, 1992](#); and references therein), or Euclidean commodity space (as in the literature on intertemporal preferences, for example [Koopmans, 1960](#)).

In contrast to those who use the full Savage model under the uncertainty interpretation, we do not rely on any extraneous assumptions about some space of outcomes to deliver the measure representation, keeping our analysis firmly focused on beliefs. In contrast to most studies of intertemporal preference, our approach allows for finite consumption spaces; we revisit this point in the [Conclusion](#).

1.3 Outline

In [Section 2](#), we provide the model and formally state our five theorems. We first define (ordinal) qualitative probability, (cardinal) quantitative probability, and representation of the former by the latter, then formally state our main result ([Theorem 5](#)). We then introduce *catalogued spaces* where each atom has a singleton core, and formally state the other four theorems.

The next four sections are dedicated to the techniques involved in the proof of [Theorem 5](#). (Structurally, [Theorem 3](#) is one of many lemmas in this proof.) In particular:

- [Section 3](#) is dedicated to the proof of [Theorem 1](#), which states that *monotone continuity* and 3-*AS* guarantee σ -measure representation for the discrete catalogued spaces, where $|S| = |\mathbb{N}|$ and each singleton is a core. The overall strategy is to construct a particular kind of order-dense family of equivalence classes called a *supercabinet*; some of the key techniques involve viewing the events as members of the Cantor set, constructing specified events using a greedy algorithm, and verifying addition axioms similar to Peano’s for a particular binary operation.
- [Section 4](#) is dedicated to the proof of [Theorem 2](#), which states that *monotone continuity* and 3-*AS* guarantee σ -measure representation for catalogued spaces with a finite number of cores. The proof involves partitioning an infinitely-divisible event into ‘honorary cores’ and applying the techniques from the previous section.

- [Section 5](#) is dedicated to the proof of [Theorem 4](#), which states that *monotone continuity* and 3^+ -AS guarantee σ -measure representation for the remaining catalogued spaces, where there is a messy sequence of cores together with some infinitely-divisible events. One of the key lemmas is [Theorem 3](#), which states that *monotone continuity* and *unlikely cores* guarantee σ -measure representation. Constructing the σ -measure involves repeatedly splitting the space into a *left space* and a *right space*, then applying [Theorem 2](#) to the former and [Theorem 3](#) to the latter. Verifying that the σ -measure is in fact a representation involves an algorithm that takes as input an arbitrary pair of events A and B , and returns as output a finite sequence of event pairs, allowing the statement “ $A \succsim B$ if and only if $\mu(A) \geq \mu(B)$ ” to be directly proved.
- [Section 6](#) briefly describes how [Theorem 5](#) is established by the above results.

Finally, [Section 7](#) concludes by discussing two alternative settings where our results are relevant: preferences over parts of a heterogeneous good and preferences over consumption streams.

2 Model and main results

2.1 Qualitative probability

A *likelihood space* $(S, \mathcal{A}, \succsim)$ is a triple consisting of

- (1) a set of *states* S with generic members s, s' ,
- (2) a σ -algebra of *events* $\mathcal{A} \subseteq 2^S$, including S , with generic members A, B, C ,¹⁵ and
- (3) a *likelihood relation* \succsim , which is a binary relation on \mathcal{A} .

We interpret \succsim as a collection of comparisons of events on the basis of relative likelihood: $A \succsim B$ denotes “ A is at least as likely as B .” We write $A \sim B$ to denote $A \succsim B$ and $B \succsim A$, and write $A \succ B$ to denote $A \succsim B$ but not $A \sim B$. We emphasize that, for convenience, we include degenerate triples $(\emptyset, \{\emptyset\}, \succsim)$ as likelihood spaces.

Formally, all our results concern likelihood spaces. That said, we often implicitly take S and \mathcal{A} as the set of states and σ -algebra, respectively, abusing notation by referring to the likelihood space by its likelihood relation. In certain cases, we are instead explicit about the entire likelihood space, particularly when comparing several likelihood spaces or when imposing structure on the set of states and σ -algebra.

The following assumptions are standard:

- ORDER:¹⁶ The relation \succsim is complete and transitive.

¹⁵[Savage \(1954\)](#) is less general, asserting that $\mathcal{A} = 2^S$, while others are more general, allowing \mathcal{A} to be any “abstract” σ -algebra ([Villegas, 1964](#); [Villegas, 1967](#)), or any algebra ([Wakker, 1981](#); [Marinacci, 1993](#)), or any mosaic ([Kopylov, 2007](#); [Abdellaoui and Wakker, 2005](#)). This last generalization is motivated by the distinction between “risk” and “ambiguity” ([Knight, 1921](#)) illustrated by the Ellsberg paradox ([Ellsberg, 1961](#)), as the collection of “subjectively risky events” ([Epstein and Zhang, 2001](#)) need only be a mosaic.

¹⁶The standard assumption of *completeness*—that comparisons are always possible—has been criticized on both normative and positive grounds ([von Neumann and Morgenstern, 1947](#); [Aumann, 1962](#); [Schmeidler, 1989](#); [Mandler, 2005](#)). Indecisiveness can be observed and distinguished from indifference ([Eliaz and Ok, 2006](#)), and in fact has been in a recent experiment ([Cettolin and Riedl, 2013](#)). Recently, likelihood relations have been considered without *completeness*, and in particular, axiomatic foundations

◦ SEPARABILITY:¹⁷ For each triple $A, B, C \in \mathcal{A}$ such that $A \cap C = B \cap C = \emptyset$,

$$A \succsim B \text{ if and only if } A \cup C \succsim B \cup C.$$

◦ MONOTONICITY: For each pair $A, B \in \mathcal{A}$, $A \subseteq B$ implies $B \succsim A$.

◦ NONDEGENERACY: There are $A, B \in \mathcal{A}$ such that $A \succ B$.

We gather these standard assumptions in the following definition:

DEFINITION: A *qualitative probability space* is a likelihood space that satisfies *order*, *separability*, *monotonicity*, and *nondegeneracy*. A *qualitative probability* is the likelihood relation of a qualitative probability space.¹⁸

2.2 Quantitative probability

We seek conditions under which our ordinal notion of qualitative probability space coincides with the standard cardinal notion of (quantitative) probability space (Kolmogoroff, 1933). The cardinal notion is that of a probability measure:

DEFINITION: A function $\mu : \mathcal{A} \rightarrow [0, 1]$ is a *measure* if

- (1) $\mu(S) = 1$, and
- (2) for each finite, pairwise disjoint collection $\{A_i\}_{i \in I} \subseteq \mathcal{A}$,

$$\mu(\cup A_i) = \sum \mu(A_i).$$

A measure μ is moreover a σ -*measure* if

- (3) for each countably infinite, pairwise disjoint collection $\{A_i\}_{i \in I} \subseteq \mathcal{A}$,

$$\mu(\cup A_i) = \sum \mu(A_i).$$

Let $\mathbb{M}(\mathcal{A}) \subseteq [0, 1]^{\mathcal{A}}$ denote the set of measures and let $\mathbb{M}^{\sigma}(\mathcal{A}) \subseteq \mathbb{M}(\mathcal{A})$ denote the set of σ -measures.

We say that the ordinal and cardinal notions coincide when the measure represents likelihood comparisons:

DEFINITION: If $(S, \mathcal{A}, \succsim)$ is a qualitative probability space and $\mu \in \mathbb{M}(\mathcal{A})$, then we say μ is a *representation of* $(S, \mathcal{A}, \succsim)$ if for each pair $A, B \in \mathcal{A}$,

$$A \succsim B \text{ if and only if } \mu(A) \geq \mu(B).$$

In this case we say μ *represents* \succsim .

for representation by multiple prior probability measures—where a first event is more likely than a second if and only if each prior assigns a higher probability to the first—have been provided, both when each event can be partitioned into two equally-likely subevents (Nehring, 2009) and more generally through the use of a cancellation axiom (Alon and Lehrer, 2014).

¹⁷We note that when likelihood relations are interpreted as observable preferences among bets, the systematic violation of this axiom has been observed in an experiment where certain events are attached to probability appraisals while others are not (Ellsberg, 1961); the favoring of the appraised events is a phenomenon typically ascribed to *ambiguity aversion* (though not always; see Ergin and Gul, 2009).

¹⁸Savage (1954) and others give an equivalent definition using a weaker version of *monotonicity* and a stronger version of *nondegeneracy*.

2.3 The main result

As argued in [Section 1.1](#), there are strong reasons to impose that the measure representation is σ -additive ([Adams, 1962](#); [Seidenfeld and Schervish, 1983](#); [Wakker, 1993b](#); [Stinchcombe, 1997](#)), and the appropriate axiom for doing so has been identified:¹⁹

- MONOTONE CONTINUITY ([VILLEGAS, 1964](#)): For each $A \in \mathcal{A}$ and each $(B_i) \in \mathcal{A}^{\mathbb{N}}$, if
 - (i) $B_1 \supseteq B_2 \supseteq \dots$, and
 - (ii) for each $i \in \mathbb{N}$, $B_i \succsim A$,
 then $\cap B_i \succsim A$.

THEOREM V1 ([VILLEGAS, 1964](#)): If $(S, \mathcal{A}, \succsim)$ is a qualitative probability space with representation $\mu \in \mathbb{M}(\mathcal{A})$, then $(S, \mathcal{A}, \succsim)$ satisfies *monotone continuity* if and only if $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

We therefore impose *monotone continuity*, whose implications are understood when there are no ‘atoms,’ or events whose less-likely subevents are null:

DEFINITION: An *atom* is an event $A \in \mathcal{A}$ such that

- (1) $A \succ \emptyset$, and
- (2) for each $B \subseteq A$, either $B \sim A$ or $B \sim \emptyset$.

We write $\mathcal{A}^\alpha \subseteq \mathcal{A}$ for the collection of atoms.

In particular, when there are no atoms, *monotone continuity* guarantees a unique representation by a special kind of measure:

DEFINITION: A measure $\mu \in \mathbb{M}(\mathcal{A})$ is *infinitely-divisible* if for each $A \in \mathcal{A}$ and each $\lambda \in [0, 1]$, there is $B \subseteq A$ such that $\mu(B) = \lambda\mu(A)$.

THEOREM V2 ([VILLEGAS, 1964](#)): If $(S, \mathcal{A}, \succsim)$ is a qualitative probability space satisfying *monotone continuity* such that $\mathcal{A}^\alpha = \emptyset$, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$. Moreover, μ is infinitely-divisible.

In this article, we generalize [Theorem V2](#) by allowing atoms. To do so, we introduce a parametric family of atom-swarmling axioms, with higher-parameter assumptions logically stronger than lower-parameter ones. Formally, for each $A \in \mathcal{A}$, let $(\downarrow A)$ denote the collection of non-null events that are less likely than A :

$$(\downarrow A) \equiv \{A' \in \mathcal{A} \mid A \succ A' \succ \emptyset\}.$$

For each $k \in \mathbb{N}$, we define the axiom that for each atom A , $\cup(\downarrow A)$ is, in an ordinal sense, k times as likely as A :

- k TH-ORDER ATOM-SWARMING (k -AS): For each $A \in \mathcal{A}^\alpha$, there are
 - (i) pairwise-disjoint $\mathcal{B} \subseteq (\downarrow A)$, and

¹⁹In fact, this is the appropriate axiom even in the multiple priors model ([Gilboa and Schmeidler, 1989](#)), guaranteeing that the set of priors is a relatively weak compact set of σ -measures ([Chateauneuf, Maccheroni, Marinacci, and Tallon, 2005](#)).

(ii) $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k \subseteq \mathcal{B}$ partitioning \mathcal{B} ,
such that for each $i \in \{1, 2, \dots, k\}$, $(\cup \mathcal{B}_i) \succsim A$.

If furthermore, for each $i \in \{1, 2, \dots, k\}$, $(\cup \mathcal{B}_i) \succ A$, then *strict k th-order atom-swarming*, or k^+ -AS, is satisfied.

As promised, higher-indexed axioms are logically stronger. Note that under *monotone continuity*, any pairwise-disjoint collection of non-null events \mathcal{B} is countable, so if \mathcal{B}_i is a subcollection of some such \mathcal{B} , then $(\cup \mathcal{B}_i)$ is an event.

Our main result is that σ -representation is guaranteed by *monotone continuity* and 3^+ -AS:

THEOREM 5: A triple $(S, \mathcal{A}, \succsim)$ is a qualitative probability space satisfying *monotone continuity* and 3^+ -AS if and only if it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ such that for each $A \in \mathcal{A}$,

- (i) if A is an atom, $\mu(\cup(\downarrow A)) > 3\mu(A) > 0$, and
- (ii) if A contains no atoms, for each $\lambda \in [0, 1]$, there is $B \subseteq A$ such that $\mu(B) = \lambda\mu(A)$.

2.4 Catalogued spaces and other results

Working with atoms involves some nuance: two distinct atoms can have non-empty intersection; in fact, one atom can even be contained in another. This makes it tempting to consider only ‘smallest’ atoms: for each atom, take the collection of its non-null subevents (which are themselves atoms), then take the intersection of this collection. The problem is that in general, this collection need not be countable, and therefore this intersection need not be an event.

We suppress the above nuance for most of the analysis by asserting that each atom contains a *core*, or a singleton atom, and do so by working with the following class of qualitative probability spaces:

DEFINITION: A qualitative probability space $(S, \mathcal{A}, \succsim)$ is *catalogued* if there is $S^\oplus \subseteq S$ such that

- (1) either
 - (i) $S^\oplus = \emptyset$,
 - (ii) there is $m \in \mathbb{N}$ such that $S^\oplus = \{1, 2, \dots, m\}$, or
 - (iii) $S^\oplus = \mathbb{N}$,
- (2) for each $s \in S^\oplus$, $\{s\} \in \mathcal{A}^\alpha$,
- (3) for each $A \subseteq S \setminus S^\oplus$, $A \notin \mathcal{A}^\alpha$, and
- (4) for each pair $s, s' \in S^\oplus$, $s < s'$ implies $\{s\} \succsim \{s'\}$.

We let \mathcal{A}^\oplus denote 2^{S^\oplus} , we let S° denote $S \setminus S^\oplus$, and we let \mathcal{A}° denote $\mathcal{A} \cap 2^{S^\circ}$. We say that $(S, \mathcal{A}, \succsim)$ is

- *purely- σ -catalogued* if $S^\oplus = S = \mathbb{N}$,
- *finitely-catalogued* if $S^\oplus \neq \mathbb{N}$, and
- *σ -catalogued* if $S^\oplus = \mathbb{N}$.

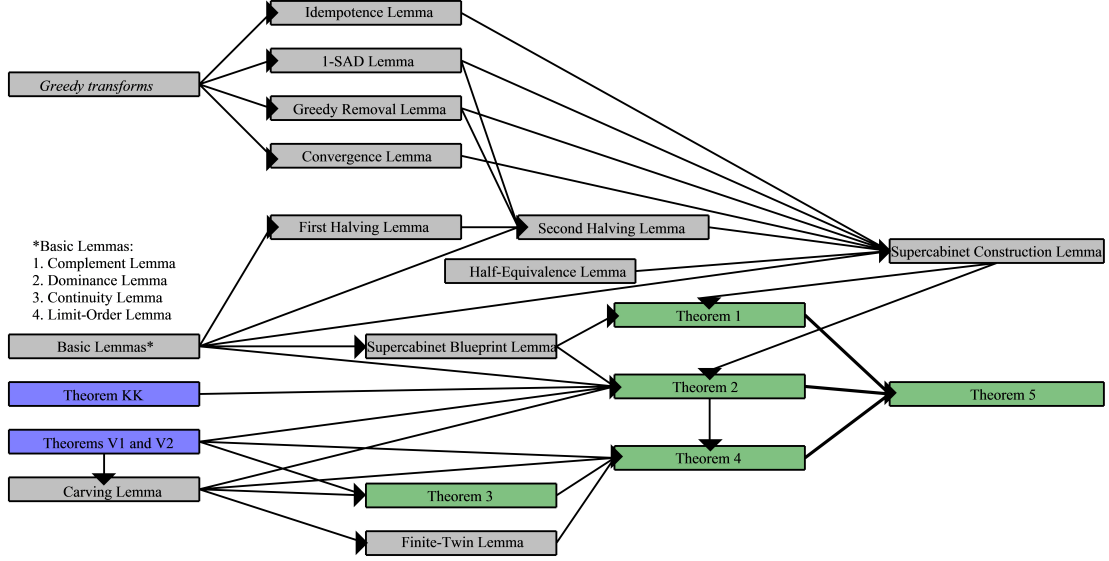


Figure 2: The results and their relationships. There is an arrow from a first result to a second if the first result is used at some point in the proof of the second. Blue results are due to others, green results are theorems, and the main result is Theorem 5.

Thus in a catalogued space, each atom has a core $\{s\}$, where $s \in S^\oplus$. Moreover, these singletons are labeled by consecutive natural numbers so that $\{1\} \precsim \{2\} \precsim \dots$, which will frequently be convenient.

We provide another generalization of [Theorem V2](#) for catalogued spaces, using the requirements that the cores are not more likely than their complement:

◦ **UNLIKELY CORES:** A catalogued probability space satisfies the axiom if $S^\circ \precsim S^\oplus$.

THEOREM 3: If $(S, \mathcal{A}, \precsim)$ is a catalogued qualitative probability space satisfying *monotone continuity* and *unlikely cores*, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

Structurally, the other three theorems are the supporting pillars of [Theorem 5](#), and the next three sections are dedicated to explaining their proofs. [Theorem 1](#), [Theorem 2](#), and [Theorem 4](#) each provide sufficient conditions for σ -measure representation of certain catalogued spaces—in particular, for purely- σ -catalogued spaces, finitely-catalogued spaces, and σ -catalogued spaces, respectively:

THEOREM 1: If $(S, \mathcal{A}, \precsim)$ is a purely- σ -catalogued qualitative probability space satisfying *monotone continuity* and 3-*AS*, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

THEOREM 2: If $(S, \mathcal{A}, \precsim)$ is a finitely-catalogued qualitative probability space satisfying *monotone continuity* and 3-*AS*, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

THEOREM 4: If $(S, \mathcal{A}, \precsim)$ is a σ -catalogued qualitative probability space satisfying *monotone continuity* and 3⁺-*AS*, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

The theorems are numbered according to the sequence they are proved in the following sections. For a schematic of the overall proof structure, see [Figure 2](#).

3 Proof sketch for Theorem 1

3.1 Supercabinet blueprint

We begin by investigating *purely σ -catalogued* probability spaces: S is the natural numbers; \mathcal{A} is its power set; and the singletons are the cores, which are labeled in non-increasing order of likelihood. For both cardinal and ordinal purposes, it is fruitful to consider \mathcal{A} with the topology of pointwise convergence: for measures, σ -additivity is equivalent to continuity (with this topology), while for qualitative probabilities, *monotone continuity* is equivalent to closure of upper and lower contour sets (with this topology).²⁰

We can thus restate our objective: we seek to prove that there is a continuous representation of \succsim that is moreover a measure. A first step might be to establish that there is *some* continuous representation, and toward this end it is helpful to recall a classic result from the utility function literature, [Theorem CD](#) ([Cantor, 1895](#); [Debreu, 1954](#); [Debreu, 1964](#)). This theorem involves a structured family of equivalence classes, which I will suggestively refer to as a *cabinet*.²¹

DEFINITION: For each qualitative probability space $(S, \mathcal{A}, \succsim)$, a collection of equivalence classes $\{\mathcal{Z}_i\}_{i \in \mathbb{N}} \subseteq [\mathcal{A}/\sim]$ is a *cabinet* of $(S, \mathcal{A}, \succsim)$ if for each pair $A, B \in \mathcal{A}$ such that $A \succ B$, there is $Z \in \cup \mathcal{Z}_i$ such that $A \succsim Z \succsim B$.

A cabinet can be thought of as a collection of drawers labeled $1, 2, 3, \dots$, each drawer filled with equally-likely events. The requirement is that if I give a file clerk two events A and B , one more likely than the other, then he should be able to go to the cabinet, open one of the drawers, and retrieve some event Z that is between A and B in likelihood. [Theorem CD](#) states that if we can construct such a cabinet, then we are promised a continuous representation:

THEOREM CD ([CANTOR, 1895](#); [DEBREU, 1954](#); [DEBREU, 1964](#)):²² If (i) $(S, \mathcal{A}, \succsim)$ has a cabinet, and (ii) \succsim has closed upper and lower contour sets, then there is continuous $u : \mathcal{A} \rightarrow \mathbb{R}$ such that

$$u(A) \geq u(B) \Leftrightarrow A \succsim B.$$

For our objective, we do not want just any continuous representation; we want one that is moreover a measure, taking into account the algebraic structure of the events. We therefore do not want just any cabinet; we want one that moreover takes into account the algebraic structure of the events. What I propose is that we first repaint the cabinet, labeling the drawers not with $1, 2, 3, \dots$, but rather in anticipation of probabilities: let v

²⁰I was unable to find a reference for either of these observations, though I would not be surprised if they are already known, especially the former.

²¹This pedagogical device was inspired by a recorded lecture of physicist Richard Feynman, where he explains how a computer can be viewed as an extremely fast but extremely stupid file clerk.

²²Cantor first proved the result under the stronger $A \succ Z \succ B$ assumption ([Cantor, 1895](#)). Nearly six decades later—at the suggestion of Savage, according to the acknowledgment—Debreu attempted to prove the result under the weaker $A \succsim Z \succsim B$ assumption ([Debreu, 1954](#)). Debreu then noticed an error in his own proof, which he corrected a decade later using his acclaimed Gap Lemma ([Debreu, 1964](#)). For narrative clarity, we have chosen a weaker statement than that in [Debreu \(1964\)](#). In fact \mathcal{A} need not be a σ -algebra, and without assumption (ii), Debreu proves there is a representation u which is upper semi-continuous in any topology for which upper contour sets are closed and lower semi-continuous in any topology for which lower contour sets are closed.

label the drawer filed with events to which we should like to eventually assign probability v . Since we can only choose a countably infinite collection of anticipated probabilities from $[0, 1]$ as labels, I propose we take the *dyadic rationals*, $\mathbb{2} \subseteq [0, 1]$:

$$\mathbb{2} \equiv \left\{ \sum_{i \in F} \left(\frac{1}{2}\right)^i \mid F \subseteq \mathbb{N}, |F| < |\mathbb{N}| \right\} \cup \{1\}.$$

The dyadic rationals can be thought of as the labels appearing anywhere in a sequence of progressively-finer rulers, the first ruler measuring halves of a foot, the second ruler measuring fourths, the third ruler measuring eighths, and in general the k th ruler measuring $(\frac{1}{2^k})$ ths.

Of course, it is not enough to simply repaint the cabinet; these new labels must be operationalized:

DEFINITION: For each qualitative probability space $(S, \mathcal{A}, \succsim)$, a collection of equivalence classes $\{\mathcal{Z}_v\}_{v \in \mathbb{2}} \subseteq [\mathcal{A}/\sim]$ is a *supercabinet* of $(S, \mathcal{A}, \succsim)$ if

[SC1] $\emptyset \in \mathcal{Z}_0$ and $S \in \mathcal{Z}_1$,

[SC2] for each pair $v, v' \in \mathbb{2}$ such that $v + v' \leq 1$, there are disjoint $Z_v \in \mathcal{Z}_v$ and $Z_{v'} \in \mathcal{Z}_{v'}$ such that $Z_v \cup Z_{v'} \in \mathcal{Z}_{v+v'}$,

[SC3] for each non-increasing $(v_i^+) \in \mathbb{2}^{\mathbb{N}}$ and non-decreasing $(v_i^-) \in \mathbb{2}^{\mathbb{N}}$ such that $\lim v_i^+ = \lim v_i^-$,²³ there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$, and

(ii) $\lim A_i^+ = \lim A_i^-$.

[SC4] for each monotonic pair $(v_i), (w_i) \in \mathbb{2}^{\mathbb{N}}$ such that for each $i \in \mathbb{N}$, $v_i + w_i \leq 1$, there are convergent $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$ such that for each $i \in \mathbb{N}$, A_i and B_i are disjoint with $A_i \in \mathcal{Z}_{v_i}$ and $B_i \in \mathcal{Z}_{w_i}$.

Though it is not obvious by comparing the definitions, a supercabinet is a special kind of cabinet; this follows from our next lemma. The key idea behind the additional structure is given by [SC2]: if I give a file clerk two anticipated probabilities v and v' , and their sum does not exceed 1, then he should be able to take an event Z_v from the drawer labeled v and an event $Z_{v'}$ from the drawer labeled v' . Moreover, he should be able to choose this pair so that Z_v and $Z_{v'}$ are disjoint. Finally, he should be able to find their union $Z_v \cup Z_{v'}$ filed in the drawer labeled $v + v'$. The following lemma promises that if we can construct a supercabinet, then we are done:

SUPERCABINET BLUEPRINT LEMMA: If $(S, \mathcal{A}, \succsim)$ is a qualitative probability space that has a supercabinet and satisfies *monotone continuity*, then there is a unique $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ such that μ represents \succsim .

As expected, for each $v \in \mathbb{2}$, the unique measure representation assigns to each event in \mathcal{Z}_v the probability v ; indeed by design, any measure representation must. A continuity argument takes care of the un-filed events; see [Appendix 2](#) for the proof.

With the blueprint in hand, let us begin construction.

²³Since both sequences are bounded, they are also convergent.

3.2 Constructing halves

It is easy enough to file the empty set (and those events equally likely) into \mathcal{Z}_0 , and to file the state space (and those events equally likely) into \mathcal{Z}_1 . The next drawer we shall populate is $\mathcal{Z}_{\frac{1}{2}}$. This might well be expected: such “even-chance” events play a role in the approach outlined by Ramsey (1931), and their existence is sometimes directly assumed (see for example Gul, 1992). Here, however, we need not make any such assumption; we shall instead construct such an event.

We can already compare two events according to relative likelihood with \succsim ; let us forget that for a moment. Let us instead consider comparing two events *lexicographically*:

$$A >_{LEX} B \text{ if and only if } \min A \setminus B < \min B \setminus A.$$

In other words, when comparing distinct A and B , let s be the smallest-index state for which s is in one event but not the other. If $s \in A$, then $A >_{LEX} B$, else $B >_{LEX} A$.

It is well-known that there is an order-preserving homeomorphism—let us call it Ψ —between

- (i) the subsets of the natural numbers ordered lexicographically, and
- (ii) the canonical Cantor set (Cantor, 1883) with the standard ordering of reals.²⁴

This observation is important because of the intimate relationship between order and topology on the real line—after all, the standard topology is generated by upper and lower contour sets! Indeed, on the real line each compact set has a maximum, and because the Cantor set is bounded, by the Heine-Borel Theorem each of its closed subsets has a maximum. Because of Ψ , in our environment every closed collection of events has a lexicographic maximum.

In particular, the collection of events that are no more likely than their complements,

$$\mathcal{A}^- \equiv \{B \in \mathcal{A} \mid (S \setminus B) \succsim B\},$$

is closed, and therefore has a lexicographic maximum H .

Can H have a finite complement? Under 1- AS , no: if there were some $s = \max(S \setminus H)$, then the event $(H \setminus \{s + 1, s + 2, \dots\}) \cup \{s\}$ would be lexicographically greater than H , but by 1- AS would also belong to \mathcal{A}^- , contradicting the definition of H . Thus H has an infinite complement, and by construction the union of H with any state in $(S \setminus H)$ is

- (i) lexicographically greater than H , and thus
- (ii) outside of \mathcal{A}^- , and thus
- (iii) more likely than its complement.

It follows from a continuity argument that H is just as likely as its complement, and can therefore be properly be filed into drawer $\mathcal{Z}_{\frac{1}{2}}$. In order to use the full power of this observation, we wish to state it when the event we are trying to split in halves is not necessarily all of S , but rather might be any collection A with the required structure. We do this using the notion of *subspaces*:

²⁴In fact, Ψ has a simple explanation. Consider the standard construction of the Cantor set through iterative removal of middle thirds. A point in the Cantor set is identified through a sequence of ‘lefts’ and ‘rights’: at each stage of the construction, we record some smallest interval, and after middle thirds are removed, we record either its left or right subinterval; the point is the unique intersection of this sequence of recorded intervals. For each $A \subseteq \mathbb{N}$, $\Psi(A)$ chooses ‘right’ at the k th step if $k \in A$; otherwise it chooses ‘left’—in this way a sequence of inclusions and exclusions encodes a sequence of rights and lefts to uniquely determine a point in the Cantor set. By inspection, Ψ is an order-preserving bijection, and it is routine to show that Ψ and its inverse are continuous.

DEFINITION:²⁵ For each likelihood space $(S, \mathcal{A}, \succsim)$ and each $A \in \mathcal{A}$, define the A -subspace of $(S, \mathcal{A}, \succsim)$, $(S, \mathcal{A}, \succsim)|_A$, to be the likelihood space $(A, \mathcal{A} \cap 2^A, \succsim|_A)$, where $\succsim|_A$ is the binary relation on $\mathcal{A} \cap 2^A$ defined by:

$$B \succsim|_A C \text{ if and only if } B, C \in \mathcal{A} \cap 2^A \text{ and } B \succsim C.$$

We sometimes abuse notation, writing $\succsim|_A$ to denote the entire likelihood space $(S, \mathcal{A}, \succsim)|_A$. Using this notion, we can talk about ‘halving’ events with the required structure:

FIRST HALVING LEMMA: If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *monotone continuity*, then for each $A \in \mathcal{A}$ such that $A \succ \emptyset$ and $\succsim|_A$ satisfies 1-AS, there is $H \subseteq A$ such that $H \sim A \setminus H$ and $A \succ H \succ \emptyset$.

The proof, sketched throughout this section, is in [Appendix 4](#). Thus it is through the Cantor set that we first glimpse cardinality in our ordinal relation.

3.3 Greedy transforms

In practice, what must come next is the population of $\mathcal{Z}_{\frac{1}{4}}$, then $\mathcal{Z}_{\frac{1}{8}}$, and so on. For the moment, however, let us set this problem aside and consider another: if we have populated drawers \mathcal{Z}_v and $\mathcal{Z}_{v'}$, with $v + v' \leq 1$, how can we take one event from \mathcal{Z}_v and another event from $\mathcal{Z}_{v'}$ so that these two events do not intersect?

Remarkably, it turns out that regardless of v' , we can always succeed if we first ‘blindly’ take a specific event from \mathcal{Z}_v . Moreover, this event is specifically the lexicographic-maximum of \mathcal{Z}_v . Moreover, this event can be constructed with a so-called “greedy” algorithm. This section is dedicated to that algorithm, which turns out to have further use in the construction of the supercabinet.

In order to build intuition, it is helpful to once again anticipate probabilities—especially, here, as numbers associated with singleton events. For the sake of analogy, let us think of singleton events as coins and the associated numbers as monetary values. Let us also think of v as a bus fare, and consider the following scenario:

SCENARIO 1: You would like to take a bus, and in order to do so, you must pay the fare with exact change. The fare is described to you with a sign specifying the number v . In your wallet, you have various coins in different denominations. Can you make exact change, and if so, how?

One possible approach is to use the following greedy algorithm: consider your coins, one at a time, in non-increasing order of value. If the coin under consideration puts your running total over the fare, then return it to your wallet; otherwise, place it into your palm. Once you have considered every coin, present the pile in your palm to the bus driver.

²⁵We emphasize that because $\succsim|_A$ compares only subevents of A , this notion is distinct from conditional qualitative probability ([Savage, 1954](#)). We do not use conditional probabilities anywhere in this article, and we postulate nothing about how beliefs change in response to new information.

It is easy to see the greedy algorithm can unnecessarily fail, for example if the fare is 30¢ when you have a quarter and three dimes.²⁶ But there is a particular class of problems for which the greedy algorithm always works: when you have a countably infinite collection of coins whose value tends to zero, your total wealth exceeds the fare, and the value of each coin is less than the value of all smaller coins.²⁷ Essentially, the algorithm works because there is always enough and never too much: if a coin is skipped, there is always enough to make up for it, and by design, there is never too much in your palm.

For our purposes, this is only the intuition: while the above logic will apply when at last we have probabilities, we do not have them yet. Right now, we only have some populated drawer \mathcal{Z}_v ; there may not be any singletons at all that have yet been filed! Nevertheless, it is possible to modify the analogy to suit our situation by considering another world where coins do not feature numbers, but rather the value of a pile of coins is given by its mass:

SCENARIO 2: You would like to take a bus, and in order to do so, you must pay the posted fare with exact change. The fare is described to you with a pile of coins B on one arm of a balance. In your wallet, you have various coins. Can you make exact change, and if so, how?

As it turns out, under the analogous ordinal condition for the coins in your wallet, an analogue of the greedy algorithm works. If A represents the coins in your wallet, and B represents the fare, then let $\mathcal{G}^A(B)$ represent the outcome of the greedy algorithm:

DEFINITION: If $(S, \mathcal{A}, \succsim)$ is purely- σ -catalogued, then for each $A \in \mathcal{A}$, we define the *greedy transform* $\mathcal{G}^A : \mathcal{A} \rightarrow \mathcal{A}$ as follows. For each $B \in \mathcal{A}$, define $\mathcal{G}^A(B) \subseteq A$ by:

- Define $\mathcal{G}_0^A(B) \equiv \emptyset$.
- For each $i \in \mathbb{N}$, define

$$\mathcal{G}_i^A(B) \equiv \begin{cases} \mathcal{G}_{i-1}^A(B) \cup \{i\}, & i \in A \text{ and } B \succsim \mathcal{G}_{i-1}^A(B) \cup \{i\} \\ \mathcal{G}_{i-1}^A(B), & \text{else.} \end{cases}$$

- Clearly $(\mathcal{G}_i^A(B)) \in \mathcal{A}^{\mathbb{N}}$ is convergent. Define

$$\mathcal{G}^A(B) \equiv \lim \mathcal{G}_i^A(B).$$

By construction and by *monotone continuity*, $B \succsim \mathcal{G}^A(B)$. Provided $A \succsim B$, when is equivalence guaranteed? As promised:

1-AS LEMMA: If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *monotone continuity*, then for each $A \in \mathcal{A}$ such that $\succsim|_A$ satisfies 1-AS and each $B \in \mathcal{A}$ such that $A \succsim B$, $\mathcal{G}^A(B) \sim B$.

²⁶This failure is an implication of a broader principle: determining whether or not you can make exact change is *NP*-complete, even if it is given that all coin values and the fare are integers! This is the *subset sum problem*, a special case of the *knapsack problem*; see for example [Garey and Johnson \(1979\)](#).

²⁷This observation is due to Kakeya ([Kakeya, 1914](#); [Kakeya, 1915](#)), who moreover showed that for any sequence $(\mu_i) \in \mathbb{R}^{\mathbb{N}}$ with finite sum, the collection of subsequence sums is convex if and only if for each $i \in \mathbb{N}$, $\sum_{j>i} \mu(j) \geq \mu(i)$.

In other words, if A represents your collection of coins, its subspace satisfies 1- AS , and you have at least enough for the fare, then $\mathcal{G}^A(B)$ precisely pays for the fare. The intuition for why this works is the same as before—after all, “always enough” and “never too much” are ordinal statements!

Thus the file clerk can open the populated drawer \mathcal{Z}_v , remove any event B , and set it as the fare. Then, using the greedy algorithm, he can produce $\mathcal{G}^S(B)$, which by construction is necessarily the lexicographic-maximum of \mathcal{Z}_v .

3.4 Iterative halving and disjoint pairs

In the last section, we established that under 1- AS ,

(i) the lexicographic-maximum of a drawer, and
(ii) the greedy transform, with S as the wallet, of any event from that drawer,
are equivalent. But why bother? If we simply intend to take the lexicographic-maximum, why spend time on this detour about how it can be constructed?

The reason is that the greedy transforms have further use; in fact, [Appendix 3](#) is dedicated to its properties. To sketch our proof, we will discuss only the most fundamental of these properties:²⁸

GREEDY REMOVAL LEMMA: If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *monotone continuity*, then for each $k \in \mathbb{N}$, each $A \in \mathcal{A}$ such that $\succsim|_A$ satisfies $(k+1)$ - AS , and each $B \in \mathcal{A}$,

$$\succsim|_{A \setminus \mathcal{G}^A(B)} \text{ satisfies } k\text{-}AS.$$

That is, if (the subspace associated with) A satisfies $(k+1)$ - AS , then removing an image of its greedy transform yields a subevent that (has a subspace which) satisfies k - AS . As a first corollary, we have:

SECOND HALVING LEMMA: If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *monotone continuity* and 2- AS , then for each $A \in \mathcal{A}$ such that $A \succ \emptyset$, there are disjoint $H(A), H'(A) \in \mathcal{A}$ such that

- (1) $A \sim H(A) \cup H'(A)$, and
- (2) $A \succ H(A) \sim H'(A) \succ \emptyset$.

The idea is simple: remove $\mathcal{G}^S(S \setminus A)$, then apply the [First Halving Lemma](#). In this way, we can make ‘halves’ of any event, and therefore populate $\mathcal{Z}_{\frac{1}{4}}, \mathcal{Z}_{\frac{1}{8}}$, and so on. The proof is in [Appendix 4](#).

As a *second* corollary to the [Greedy Removal Lemma](#), we can construct disjoint pairs. This can be understood in terms of a third bus scenario:

SCENARIO 3: You would like to take two consecutive buses: the first to the city center, and the second from there to your destination. In order to do so, for each bus, you must pay the posted fare. The first fare is described to you with a pile of coins B_1 on one arm of a balance on the first bus; the second fare is described with a pile of coins B_2 on one

²⁸In private communication, Vilmos Komornik proved the cardinal analogue of the [Greedy Removal Lemma](#) (Komornik, 2015). [Theorem KK](#), which we use in our proof of [Theorem 2](#), is a joint implication of the results of Kakeya and Komornik.

arm of a balance on the second bus. In your wallet, you have various coins. Can you make exact change, and if so, how?

Suppose you begin with S in your wallet. As before, you can use the greedy transform to make exact change with the pile $\mathcal{G}^S(B_1)$. At that point, what remains in your wallet is $S \setminus \mathcal{G}^S(B_1)$. By the [Greedy Removal Lemma](#), the required wallet structure for the greedy algorithm to work has been preserved! Thus you can make exact change for the second bus with the pile $\mathcal{G}^{S \setminus \mathcal{G}^S(B_1)}(B_2)$. These two payments are disjoint: physically, $\mathcal{G}^S(B_1)$ remains on the first bus, while $\mathcal{G}^{S \setminus \mathcal{G}^S(B_1)}(B_2)$ remains on the second; there is no intersection between the two.

Thus given populated \mathcal{Z}_v and $\mathcal{Z}_{v'}$, the file clerk can first take some arbitrary $A \in \mathcal{Z}_v$ and $B \in \mathcal{Z}_{v'}$. These may intersect; nevertheless, the file clerk can then take Z_v to be $\mathcal{G}^S(A)$ and take $Z_{v'}$ to be $\mathcal{G}^{S \setminus Z_v}(B)$, and these do not intersect.

3.5 Supercabinet construction

We are now ready to construct the supercabinet. Let us begin with too many drawers: there is not only one labeled $\frac{1}{2}$, but also one labeled $\frac{2}{4}$, another labeled $\frac{16}{32}$, and so on. In general, for each pair $p, q \in \{0, 1, 2, \dots\}$ with $p \leq 2^q$, there is a drawer labeled $\frac{p}{2^q}$. We let \mathcal{Z}_q^p denote the drawer labeled $\frac{p}{2^q}$, and once \mathcal{Z}_q^p is populated, we let A_q^p denote its lexicographic-maximum.

First, we fill the drawers of the form $\frac{1}{2^q}$ through iterative applications of the [Second Halving Lemma](#). Then, for each q , we fill the drawers of the form $\frac{p}{2^q}$ through iterative disjoint union involving A_q^1 . To be more explicit about this latter operation: given A_q^p , we define

$$A_q^{p+1} \equiv \mathcal{G}^S(A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1)).$$

This process can be understood as the construction of a sequence of progressively finer rulers ([Figure 3](#)).

At this point, with 2-AS, it is possible to show that the extra drawers are in fact redundant: if two drawers bear labels that share a reduced fraction, then their contents are the same. So should we discard all the redundant drawers—say, keeping only those whose labels are reduced fractions—to yield our supercabinet? Not yet; the problem is that while this would yield properly-painted drawers that are populated, we do not yet know that this structure would indeed be supercabinet. In particular, we know that if we give the file clerk v and v' with $v + v' \leq 1$, he can locate Z_v in \mathcal{Z}_v and $Z_{v'}$ in $\mathcal{Z}_{v'}$ such that Z_v and $Z_{v'}$ are disjoint, but we do not know that he can locate $Z_v \cup Z_{v'}$ in $\mathcal{Z}_{v+v'}$.

Toward this end, it turns out that it is convenient to continue with the redundant drawers in order to emphasize the different constructions that lead to the same equivalence class. In particular, for each $q \in \{0, 1, \dots\}$ and each pair $p, p' \in \{0, 1, \dots, 2^q\}$ such that $p + p' \leq 2^q$, define:

$$A_q^p \uplus A_q^{p'} \equiv \mathcal{G}^S(A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'})).$$

In order to prove that this binary operation in fact captures addition—that $A_q^p \uplus A_q^{p'} = A_q^{p+p'}$ —we follow Peano's axiomatization of addition ([Peano, 1889](#)). The essential step of the inductive argument is establishing that $A_q^{p+p'} = A_q^p \uplus A_q^{p'}$ implies $A_q^{p+(p'+1)} =$

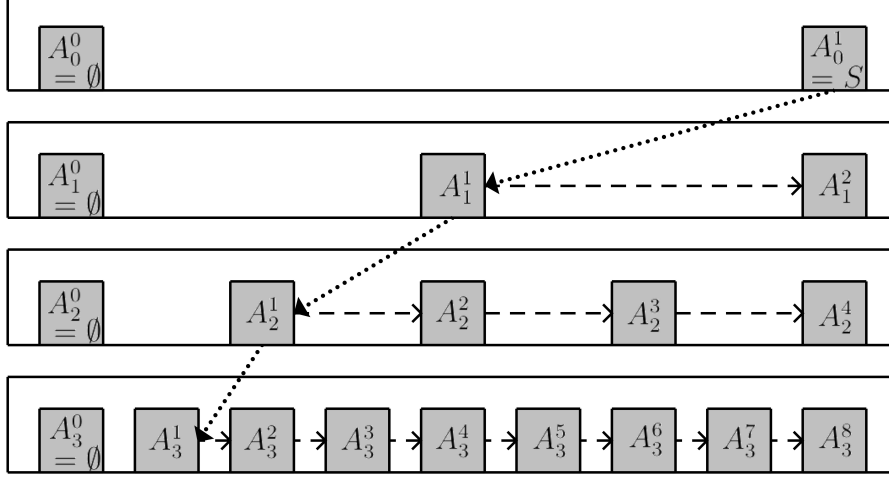


Figure 3: Each row is one member in our sequence of progressively finer rulers. The leftmost point on each is identified by \emptyset . The diagonal dotted lines represent construction through “halving.” The horizontal dashed lines represent construction through “addition by one.” It can be proved that any two events aligned vertically are in fact the same event.

$A_q^p \uplus A_q^{p'+1}$. It is here that we require 3- AS : in general, $|\{p, p', 1\}| = 3$. It is in this way that we can deduce cancellation instead of assuming it, and prove:

SUPERCABINET CONSTRUCTION LEMMA: If $(S, \mathcal{A}, \succsim)$ is purely- σ -catalogued and satisfies 3- AS , then $(S, \mathcal{A}, \succsim)$ has a supercabinet.

For details, and to see that the other supercabinet properties can be verified, see [Appendix 5](#). Together with the [Supercabinet Blueprint Lemma](#), we have at once:

THEOREM 1: If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *monotone continuity* and 3- AS , then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

The formal proof is in [Appendix 5](#).

4 Proof sketch for Theorem 2

Next, we turn our attention to *finitely-catalogued* qualitative probability spaces. As with [Theorem 1](#), we proceed by constructing a supercabinet.

Recall that S^\oplus gathers the states whose singletons are the cores; S° gathers the other states; and \mathcal{A}° is the collection of events contained in S° , which includes no atoms. Our approach is to apply the [Supercabinet Construction Lemma](#) to a purely- σ -catalogued space which is derived from our finitely-catalogued space, essentially by partitioning S° into events that we temporarily pretend are cores.

To begin, finitely-catalogued spaces involve events that contain no atoms, which it is helpful to think of as cloth-like fabric that may be cut into smaller pieces of any specification. Indeed, by [Theorem V2](#), $\succsim|_{S^\circ}$ has an infinitely-divisible measure representation

μ° , and moreover, this intuition is even harmonious with atoms:

CARVING LEMMA: If $(S, \mathcal{A}, \succsim)$ is a catalogued qualitative probability space satisfying *monotone continuity*, then for each pair $A, B \in \mathcal{A}$ such that $A \in \mathcal{A}^\circ$ and $A \succsim B$, there is $B' \subseteq A$ such that $B' \in \mathcal{A}$ and $B' \sim B$.

The proof, which unsurprisingly involves retracing some arguments from [Villegas \(1964\)](#), is in [Appendix 6](#).

How do we partition S° ? Before we begin, let us call S° the *fabric*. At each step, we cut off a piece of the fabric and reclassify it as an *honorary core*, and after an infinite sequence of steps we finish and are out of fabric. Because it is convenient to work with μ° , we first use the [Carving Lemma](#) and [Theorem V2](#) to cut off an arbitrary non-null event that is less likely than the least-likely core. By 3-AS, this first honorary core is less likely than S° as well, and we are therefore left with some (non-null) fabric. We then proceed recursively.

At the start of each step, some (non-null) fabric remains. Let A be the least-likely of the actual and honorary cores, which in fact will always be honorary. We use [Theorem V2](#) to partition what remains of the fabric into four equally likely events: B_1, B_2, B_3 , and B_4 . If $A \succsim B_1$, then we cut off B_1 ; otherwise, we use the [Carving Lemma](#) to cut off an event that is as likely as A_i . In both cases, fabric indeed remains after the cut.

Appealing to μ° , it is clear that we first cut off some finite number of pieces that have the same measure as the first honorary core, after which point each new honorary core is less likely than its predecessors. By construction, each of these latter honorary cores has a measure equal to three times the sum of the measures of its successors. Put another way, beyond a certain point, the measures of the honorary cores are geometric: if one has measure v , then the next has measure $\frac{3v}{4}$.

We now have a coarser σ -algebra $\mathcal{A}' \subseteq \mathcal{A}$: the cores, the honorary cores, and their unions. By taking the restriction of \succsim to \mathcal{A}' and relabeling the singletons, we construct a finitely-catalogued qualitative probability space that (we verify) satisfies 3-AS. At this point, we apply the [Supercabinet Construction Lemma](#) to produce a supercabinet, and in doing so reveal a supercabinet for the original space. By the [Supercabinet Blueprint Lemma](#), we have established:

THEOREM 2: If $(S, \mathcal{A}, \succsim)$ is a finitely-catalogued qualitative probability space satisfying *monotone continuity* and 3-AS, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

The formal proof is in [Appendix 6](#).

5 Proof sketch for Theorem 4

5.1 Splitting the atoms

At last, we are ready to consider σ -catalogued qualitative probability spaces, where there is some messy sequence of cores S^\oplus together with some infinitely-divisible fabric S° . Essentially, there are two major steps: (i) construct μ , and (ii) verify that it is a representation. In a way, these steps are intertwined: we do not construct μ as directly as possible, but rather take an indirect path that reveals much of what μ represents. This section is dedicated to describing this indirect path at a high level.

The fundamental technique in this construction is splitting the atoms into a *left space* and a *right space*. In particular, imagine that the cores are arranged along a line from left to right in non-increasing order of likelihood. In order to split the atoms, simply pick an adjacent pair of cores and build a wall between them. So long as this wall is in place, the left atoms are segregated from the right: we will consider the left cores with the fabric, and separately we will consider the right cores with the fabric, but never will we consider the left and right cores together. Once we are done, we may tear down that wall and build another elsewhere, but at any given time there will always be at most one wall.

Formally, for each ‘middle’ $m \in S^\oplus$, we define two qualitative probability spaces, one for the left and the other for the right:

- Define $S^{\leftarrow m} \equiv \{1, 2, \dots, m\} \cup S^\circ$; define $\mathcal{A}^{\leftarrow m}$ to be the events contained in $S^{\leftarrow m}$; and let $\succsim^{\leftarrow m}$ be the associated restriction of \succsim .
- Define $S^{m \rightarrow} \equiv \{m, m+1, \dots\} \cup S^\circ$; define $\mathcal{A}^{m \rightarrow}$ to be the events contained in $S^{m \rightarrow}$; and let $\succsim^{m \rightarrow}$ be the associated restriction of \succsim .

In the informal description, building a wall means separately considering $S^{\leftarrow m}$ and $S^{m+1 \rightarrow}$ (and their associated spaces).

Our first order of business is to establish that far enough in the tail—that is, for sufficiently high m —the left and right spaces each have a unique measure representation. For such a pair $S^{\leftarrow m}$ and $S^{m+1 \rightarrow}$, we can then stitch the two measures together, thus constructing a grand measure μ^m over the entire space \mathcal{A} that represents both the left and right spaces. Finally, we establish that all of these grand measures coincide, and we take this common measure as μ , completing the construction. (Note that there is then still more to be done, as it is not enough for a measure to represent all comparisons where both events come from the same left or right space deep in the tail.)

5.2 Deep left spaces

We first show that each sufficiently deep left space has a unique measure representation. One might view all of our work thus far as culminating in this fact: the left spaces are finitely-catalogued, and we will therefore apply [Theorem 2](#), whose proof required the construction of a supercabinet. However, we can not apply [Theorem 2](#) just yet: while left spaces are finitely-catalogued, they need not satisfy 3-*AS*.

In fact, while it is true (and we will establish) that sufficiently deep left spaces do satisfy 3-*AS* when the grand space satisfies 3⁺-*AS*, this is not necessarily so when the grand space only satisfies 3-*AS*. Indeed, suppose the most-likely core $\{1\}$ is such that $S \setminus \{1\}$ can be partitioned into three events that are each precisely as likely as $\{1\}$. In this case, each left space includes $\{1\}$, yet discards some (non-null) cores and thus cannot satisfy 3-*AS*.

As promised, this is no problem when the grand space satisfies 3⁺-*AS*. For each $s \in S^\oplus$, let us say that $t \geq s$ is a *truncation point of s* if, informally, even after truncating the tail beyond t , $\{s\}$ is third-order swarmed by less-likely non-null events. By a continuity argument, we can construct for each s a particular truncation point $\tau(s)$. Moreover, by another continuity argument, deep enough in the tail each s is its own truncation point, as eventually the cores become smaller than ‘thirds’ of the fabric. Letting $s^* \in S^\oplus$ denote the earliest s which is its own truncation point, we define

$$m^{\leftarrow*} \equiv \max\{\tau(1), \tau(2), \dots, \tau(s^* - 1), \tau(s^*)\}.$$

By construction, if $m \geq m^{\leftarrow*}$, then $(S^{\leftarrow m}, \mathcal{A}^{\leftarrow m}, \succsim^{\leftarrow m})$ satisfies 3- AS , and thus by [Theorem 2](#) has a unique representation $\mu^{\leftarrow m}$. Altogether, each sufficiently deep left space has a unique measure representation.

5.3 Deep right spaces

While each left space has a finite number of cores and some fabric, and some of them satisfy the requirements of [Theorem 2](#), this is not the case for the right spaces. Each right space is a messy sequence of cores together with some fabric, and in this way is like the grand space.

That said, in deeper right spaces the relative likelihood of the cores is smaller, and in sufficiently deep right spaces the cores are less likely than the fabric. It turns out that this is sufficient for our purposes:

THEOREM 3: If $(S, \mathcal{A}, \succsim)$ is a catalogued qualitative probability space satisfying *monotone continuity* and *unlikely cores*, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

While this result is of independent interest, for the purposes of this section [Theorem 3](#) is simply a lemma for [Theorem 4](#). It turns out that the smaller proof-within-a-proof has the same structure as the larger proof—construct the measure, then verify it is a representation—and moreover, the smaller proof’s verification step is a simpler version of the larger proof’s.

The construction of the measure is simple: use [Theorem V2](#) to construct a representation of the fabric; use the [Carving Lemma](#) to associate each union of cores with an equally likely fabric event; assign to each union of cores the measure of its associate; complete the measure and normalize.

For verification, the idea is that for each pair of events A and B , we use a particular algorithm to construct a list of pairs $(A, B) = (A_1, B_1), (A_2, B_2), (A_3, B_3), \dots$ such that for each i , we can establish:

- (i) $A_i \succsim B_i$ if and only if $A_{i+1} \succsim B_{i+1}$, and
- (ii) $\mu(A_i) \geq \mu(B_i)$ if and only if $\mu(A_{i+1}) \geq \mu(B_{i+1})$.

Moreover, the algorithm is guaranteed to terminate with a pair (A_Ω, B_Ω) for which we can prove the statement “ $A_\Omega \succsim B_\Omega$ if and only if $\mu(A_\Omega) \geq \mu(B_\Omega)$.” Visually, it can be helpful to imagine a “logical ladder” whose rails and rungs are made of if-and-only-if arrows.

How does this algorithm work? At each step i , let $n_i^\circ \in \{0, 1, 2\}$ count how many of A_i and B_i intersect S° , and similarly let $n_i^\oplus \in \{0, 1, 2\}$ count how many of A_i and B_i intersect S^\oplus . We proceed by performing the operation for the earliest of these cases which applies:

1. If $A_i = \emptyset$ or $B_i = \emptyset$, then it is easy to show that we are done.
2. If either count is zero, then we are either comparing two fabric events or two unions of cores, and it is easy to show that we are done.
3. If $n_i^\circ = 2$, then we can use the [Carving Lemma](#) to replace the smaller fabric component with one that is contained within the larger, then subtract this from both. We begin the next step with $n_{i+1}^\circ < 2$ and $n_{i+1}^\oplus = n_i^\oplus$.

4. If $n_i^\circ = 1$ and $n_i^\oplus = 2$, then one of the events is a union of cores. Because the fabric is more likely than the cores, by the [Carving Lemma](#), we can replace the union of cores with an equally-likely fabric event. We begin the next step with $n_i^\circ = 2$ and $n_{i+1}^\oplus = 1$.
5. If $n_i^\circ = 1$ and $n_i^\oplus = 1$, and neither A_i nor B_i is \emptyset , then as in the previous case, we can replace the union of cores with an equally-likely fabric event. We begin the next step with $n_i^\circ = 2$ and $n_{i+1}^\oplus = 0$.

It is clear that this process must terminate. For details, see [Appendix 7](#).

With [Theorem 3](#) established, we return to the deep right spaces in our proof of [Theorem 4](#). Indeed, we simply define

$$m^{*\rightarrow} \equiv \min \{s \in S^\oplus \mid S^\circ \precsim \{s, s+1, \dots\}\},$$

which is well-defined by a continuity argument. By construction, if $m \geq m^{*\rightarrow}$, then $(S^{m\rightarrow}, \mathcal{A}^{m\rightarrow}, \precsim^{m\rightarrow})$ satisfies *unlikely cores*, and thus by [Theorem 3](#) has a unique representation $\mu^{m\rightarrow}$. Altogether, each sufficiently deep right space has a unique measure representation.

5.4 Constructing the measure

In this section, we construct μ not once or twice, but an infinite number of times. In particular, for each sufficiently deep m , we use the results of the previous sections to construct a unique measure μ^m that represents both the associated left space and the associated right space. We then show that all of these measures are in fact the same measure, which we call μ .

To begin, define $m^* \equiv \max\{m^{\leftarrow*}, m^{*\rightarrow}\}$, take $m \geq m^*$, and split the atoms by building a wall between m and $m+1$. By the previous sections, we have unique representations $\mu^{\leftarrow m}$ and $\mu^{m+1\rightarrow}$ for the left space and the right space, respectively. To build a grand measure μ^m such that

- (i) the restriction to the left space is a positive multiple of $\mu^{\leftarrow m}$, and
- (ii) the restriction to the right space is a positive multiple of $\mu^{m+1\rightarrow}$,

it suffices to simply solve a system of equations to determine $\mu^m(S^\circ)$, $\mu^m(\{1, 2, \dots, m\})$, and $\mu^m(\{m+1, m+2, \dots\})$, then complete the measure. Altogether, μ^m exists as the unique measure on \mathcal{A} that represents both comparisons for pairs in $\mathcal{A}^{\leftarrow m}$ and for pairs in $\mathcal{A}^{m+1\rightarrow}$.

We claim that for each pair $m, m' \geq m^*$, $\mu^m = \mu^{m'}$. It suffices to prove that for each $m \geq m^*$, $\mu^m = \mu^{m+1}$, and due to the uniqueness of each measure, it suffices to prove that μ^{m+1} represents both comparisons for pairs in $\mathcal{A}^{\leftarrow m}$ and for pairs in $\mathcal{A}^{m+1\rightarrow}$. Representation for the former comparisons is immediate as $\mathcal{A}^{\leftarrow m} \subseteq \mathcal{A}^{\leftarrow m+1}$, so we need only establish representation for comparisons of pairs in $\mathcal{A}^{m+1\rightarrow}$. The argument involves establishing that normalizations of μ^m and μ^{m+1} restricted to $\mathcal{A}^{m+1\rightarrow}$ are both measure representations when further restricted to $\mathcal{A}^{m+2\rightarrow}$, and thus agree everywhere but $\{m+1\}$; and then also there as well, since by the [Carving Lemma](#) $\{m+1\}$ has the same measure as some equally-likely fabric event. For details, see Step 6 in [Appendix 8](#).

Altogether, we have a measure μ on \mathcal{A} that represents all sufficiently deep left spaces and all sufficiently deep right spaces. At this point, all that remains is to verify that μ in fact represents all comparisons.

5.5 Verification of representation

Again, we proceed as we did in the proof of [Theorem 3](#) to verify representation. In particular: again, we take an arbitrary pair of events A and B and construct a “logical ladder” of if-and-only-if arrows. Again, by algorithm, we construct a list of pairs $(A, B) = (A_1, B_1), (A_2, B_2), (A_3, B_3), \dots$ such that for each i , we can establish:

- (i) $A_i \succsim B_i$ if and only if $A_{i+1} \succsim B_{i+1}$, and
- (ii) $\mu(A_i) \geq \mu(B_i)$ if and only if $\mu(A_{i+1}) \geq \mu(B_{i+1})$.

Again, the algorithm is guaranteed to terminate with a pair (A_Ω, B_Ω) for which we can prove the statement “ $A_\Omega \succsim B_\Omega$ if and only if $\mu(A_\Omega) \geq \mu(B_\Omega)$.” But this time, the algorithm involves a partition of the state space into three parts instead of two.

Before we begin, we require one final technique. For a given core $\{s\}$, we would like to construct a *finite twin*: an equally-likely event which, if it includes any cores, only includes a finite number of less-likely ones. If the fabric alone is at least as likely as $\{s\}$, then we can do so by the [Carving Lemma](#). In fact, even if the fabric and less-likely cores are together more likely than $\{s\}$, we can succeed—provided the fabric is at least as likely as the less-likely cores:

FINITE-TWIN LEMMA: If $(S, \mathcal{A}, \succsim)$ is a σ -catalogued qualitative probability space satisfying *monotone continuity* and 1^+ - AS , then for each $s \in S^\oplus$ such that $S^\circ \succsim \{s+1, s+2, \dots\}$, there is $T_s \in \mathcal{A}$ such that

- (i) $T_s \subseteq \{s+1, s+2, \dots\} \cup S^\circ$,
- (ii) $T_s \cap S^\oplus$ is finite, and
- (iii) $T_s \sim \{s\}$.

How does the algorithm work? To begin, we partition S into S° , $S^\leftarrow \equiv \{1, 2, \dots, m^*\}$, and $S^\rightarrow \equiv \{m^*+1, m^*+2, \dots\}$. Let us say that each event $A \in \mathcal{A}$ has three components: its *left component* $A \cap S^\leftarrow$, its *right component* $A \cap S^\rightarrow$, and its *fabric component* $A \cap S^\circ$.

At each step i , we count components. More precisely, let $n_i^\circ \in \{0, 1, 2\}$ count how many of A_i and B_i intersect S° , let $n_i^\leftarrow \in \{0, 1, 2\}$ count how many of A_i and B_i intersect S^\leftarrow , and let $n_i^\rightarrow \in \{0, 1, 2\}$ count how many of A_i and B_i intersect S^\rightarrow . We proceed by performing the operation for the earliest of these cases which applies:

1. If $A_i = \emptyset$ or $B_i = \emptyset$, then it is easy to show that we are done.
2. If any count is zero, we are done. This is immediate if both A_i and B_i belong to the same left space or the same right space, while if both are contained in S^\oplus this can be established by using the [Carving Lemma](#) to exchange each event’s right component with an equally-likely fabric component, leaving us with two events in the same left space.
3. If $n_i^\circ = 2$, then we can use the [Carving Lemma](#) to replace the smaller fabric component with one that is contained within the larger, then subtract this from both. We begin the next step with $n_{i+1}^\circ < 2$ and $n_{i+1}^\rightarrow = n_i^\rightarrow$.
4. If $n_i^\circ = 1$ and $n_i^\rightarrow = 2$, then one of the events has no fabric component. Because the fabric is more likely than the cores, by the [Carving Lemma](#), we can replace this event’s right component with an equally-likely fabric component. We begin the next step with $n_i^\circ = 2$ and $n_{i+1}^\rightarrow = 1$.

5. If $n_i^\circ = 1$ and $n^\rightarrow = 1$, and neither event is only a left component, then as in the previous case, there is an event with no fabric component, and we can replace its right component with an equally-likely fabric component. We begin the next step with $n_i^\circ = 2$ and $n_{i+1}^\rightarrow = 0$.
6. If $n_i^\circ = 1$ and $n^\rightarrow = 1$, and one of the events is only a left component but neither event is \emptyset , then at this point we modify the left-only event. In particular, we use the [Finite-Twin Lemma](#) to replace the highest-index core with its finite twin. How do we know that μ assigns both twins the same measure? We know this because while the finite twin may extend far past m^* into S^\rightarrow , nevertheless, there is some deeper left space which both twins share.

It is straightforward to verify that we cannot indefinitely loop through the first four instructions, while immediately after the fifth the algorithm will terminate. Thus in order for the algorithm to loop forever, the sixth operation must be performed an infinite number of times. But this is impossible: each time the sixth operation is performed, one of the two left components becomes lexicographically-smaller; and none of the other operations alter left components; thus because \mathcal{A}^\leftarrow is finite, the process must terminate.

This completes our verification that μ is in fact a measure; for details, see [Appendix 8](#). Altogether, we have established:

THEOREM 4: If $(S, \mathcal{A}, \succsim)$ is a σ -catalogued qualitative probability space satisfying *monotone continuity* and 3^+ -AS, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

6 Proof sketch for Theorem 5

Our final objective is to establish that, as promised, there has been essentially no loss of generality in focusing on catalogued spaces where each atom has a singleton core. This last step is relatively simple: we select some pairwise-disjoint collection of atoms and remove them, and do so such that there are no atoms in what remains. In general, there is not a unique such collection, but the selection does not matter for our purposes. Using an argument of [Villegas \(1964\)](#) (though not his related result, which has a minor mistake²⁹), this collection is at most countably infinite. When we ‘pretend’ that they are cores, we can apply [Theorem 1](#), [Theorem 2](#), or [Theorem 4](#); for details, see [Appendix 9](#). This completes the proof of our main result:

THEOREM 5: A triple $(S, \mathcal{A}, \succsim)$ is a qualitative probability space satisfying *monotone continuity* and 3^+ -AS if and only if it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ such that for each $A \in \mathcal{A}$,

- (i) if A is an atom, $\mu(\cup(\downarrow A)) > 3\mu(A) > 0$, and
- (ii) if A contains no atoms, for each $\lambda \in [0, 1]$, there is $B \subseteq A$ such that $\mu(B) = \lambda\mu(A)$.

²⁹Lemma 4 of [Villegas \(1964\)](#) incorrectly claims that for each *monotonely continuous* qualitative probability, the collection of atoms is at most countably infinite. As a counterexample, consider $S = [0, 1]$ and the (*monotonely continuous*) qualitative probability on 2^S for which $\{0\} \sim S$: each event that includes 0 is an atom. To address this, we simply borrow the sound arguments and replace the faulty ones.

7 Conclusion

Our main results do not require one to interpret \mathcal{A} as a collection of events and \succsim as beliefs; we discuss two alternative interpretations here.

Preferences over parts of a heterogeneous good

What sort of economic model allows us to classify an infinitely-divisible good as either homogeneous or heterogeneous? One approach is to consider a collection of pieces into which the good may be divided, and associate each agent with a personal measure over these pieces his preferences respect. When all agents share a measure—such as gallons when the good is milk—the good is homogeneous. When the agents’ measures differ, the good is instead heterogeneous.

The classic problem of fair division, starting with [Steinhaus \(1948\)](#), is the problem of partitioning a heterogeneous good (or a *cake*) into parts (or *slices*) and then assigning those parts to agents according to some notion of fairness. Similarly, the model of land for urban economics proposed by [Berliant \(1985\)](#) treats *land* as a heterogeneous good that can be divided into *parcels*; this model has also been used in the context of fair division ([Berliant, Thomson, and Dunz, 1992](#)). The standard assumption in both settings is that each agent’s preferences are represented by a measure, and the use of atomless measures has axiomatic foundations for both preferences over slices ([Barbanel and Taylor, 1995](#)) and preferences over parcels ([Berliant, 1986](#)). Our theorem is also a preference representation theorem for these settings, differing from the existing results in that it allows atoms: *crumbs* in the context of cake-cutting,³⁰ or parcels that cannot be subdivided (such as perhaps cities or houses) in the context of land.

Intertemporal preferences

The standard model of intertemporal preference is a special case of the Savage model with an alternative interpretation of its primitives: $S = \mathbb{N}$, with its usual well-ordering, the members of S interpreted as *periods*. This literature studies preferences of an agent (or dynasty, or institution) over consumption streams, and the preferences with σ -measure subjective expected utility representations are precisely, in the language of [Olea and Strzalecki \(2014\)](#), the *time separable* class. This central class contains:

- *geometric discounting*, ([Samuelson, 1937](#); [Koopmans, 1960](#); [Koopmans, 1972](#); [Bleichrodt, Rohde, and Wakker, 2008](#)),
- *generalized hyperbolic discounting* ([Loewenstein and Prelec, 1992](#)), and
- *quasi-hyperbolic discounting* ([Laibson, 1997](#); [Olea and Strzalecki, 2014](#)).

Because this class uses atoms, the [Savage \(1954\)](#) characterization of subjective expected utility does not apply. The standard approach in the literature is to take preferences over consumption streams with a rich consumption space, such as Euclidean commodity space.

By contrast, our theorem applies in the minimal case that commodity space has two members, allowing us to handle the case where consumption in each period consists of some indivisible goods (or *objects*). This setting is also considered by [Kochov \(2013\)](#), who

³⁰An example of such a *crumb* is the small plastic figurine (*la fève*) in a French king cake. According to tradition, whomever receives it with his slice is king for the day.

proposes *patience*: consumption of the better object on dates $\{2, 3, \dots\}$ is preferred to consumption of the better object on date $\{1\}$. Kochoy establishes that if \succsim has a geometric representation and satisfies *patience*, then \succsim has no other geometric representation; a corollary is that a geometric representation is unique under 1-*AS*.

Our atom-swarmer axioms may be interpreted as stronger patience requirements—after all, each atom is a date, and for each date the less-likely non-null events (usually) are gathered by the dates that follow; thus these axioms (usually) require that at any point in time the future sufficiently overwhelms the present. In the special case that beliefs admit a geometric representation, 3⁺-*AS* is satisfied if and only if the discount factor is greater than $\frac{3}{4}$.

Appendix 1

In this appendix, we state and prove (or provide a proof reference for) four basic lemmas about qualitative probabilities: the [Complement Lemma](#), the [Domination Lemma](#), the [Continuity Lemma](#), and the [Limit-Order Lemma](#).

The [Complement Lemma](#) states that order reverses under complements. This is a slight extension of exercise 3 on page 32 of [Savage \(1972\)](#), and thus the proof is omitted.

• **COMPLEMENT LEMMA** ([SAVAGE, 1954](#)): If $(S, \mathcal{A}, \succsim)$ is a qualitative probability space, then for each $A \in \mathcal{A}$ and each pair $B, B' \subseteq A$, $B \succsim B'$ if and only if $A \setminus B' \succsim A \setminus B$.

The [Domination Lemma](#) states that for any two pairs such that the first is disjoint and dominates the second in likelihood, the union of the first is at least as likely as the union of the second. Moreover, strict domination implies the union of the first is more likely than the union of the second. This is a slight extension of exercise 5a on page 32 of [Savage \(1972\)](#), and thus the proof is omitted.

• **DOMINATION LEMMA** ([SAVAGE, 1954](#)): If $(S, \mathcal{A}, \succsim)$ is a qualitative probability space, then for each four $A, A', B, B' \in \mathcal{A}$, if

- (1) $A \cap A' = \emptyset$,
- (2) $A \succsim B$, and
- (3) $A' \succsim B'$,

then $A \cup A' \succsim B \cup B'$. Moreover, if $A \succ B$, then $A \cup A' \succ B \cup B'$.

The [Continuity Lemma](#) involves pointwise convergence, its associated topology (under which \mathcal{A} is closed), and the standard requirement that upper and lower contour sets are closed:

DEFINITION: An event sequence $(A_i) \in \mathcal{A}^{\mathbb{N}}$ is *convergent* if for each $s \in S$, either

- (1) there is $i^* \in \mathbb{N}$ such that for each $i \geq i^*$, $s \in A_i$, and
- (2) there is $i^* \in \mathbb{N}$ such that for each $i \geq i^*$, $s \notin A_i$.

In this case, we say $\bigcup_{i=1}^{\infty} (\bigcap_{j=i}^{\infty} A_j) = \bigcap_{i=1}^{\infty} (\bigcup_{j=i}^{\infty} A_j)$ is the (*pointwise*) *limit* of (A_i) , written $\lim A_i$.

◦ **CONTINUITY:** For each $A \in \mathcal{A}$ and each convergent $(B_i) \in \mathcal{A}^{\mathbb{N}}$,

- (1) if for each $i \in \mathbb{N}$, $B_i \succsim A$, then $\lim B_i \succsim A$, and

(2) if for each $i \in \mathbb{N}$, $A \succsim B_i$, then $A \succsim \lim B_i$:

Though *continuity* seems stronger than *monotone continuity*, in fact the two are equivalent for qualitative probabilities, a point which to our knowledge has not been made previously. Based on this equivalence, we are justified in writing (*monotone*) *continuity* in stating our results while using *continuity* in our proofs:

• **CONTINUITY LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a qualitative probability space, then it satisfies *monotone continuity* if and only if it satisfies *continuity*.

PROOF: Because for each sequence $(B_i) \in \mathcal{A}^{\mathbb{N}}$ such that $B_1 \supseteq B_2 \supseteq \dots$, $\cap B_i = \lim B_i$, clearly *continuity* implies *monotone continuity*. Suppose \succsim is a qualitative probability satisfying *monotone continuity*.

To see that upper contour sets are closed, let $A \in \mathcal{A}$ and let $(B_i) \in \mathcal{A}^{\mathbb{N}}$ be convergent such that for each $i \in \mathbb{N}$, $B_i \succsim A$. Define $B_{\infty} \equiv \lim B_i$. For each $i \in \mathbb{N}$, define $C_i \equiv \cup_{j \geq i} B_j$. By construction, $C_1 \supseteq C_2 \supseteq \dots$, and by *monotonicity*, for each $i \in \mathbb{N}$, $C_i \succsim B_i \succsim A$; thus by *monotone continuity*, $\cap C_i \succsim A$.

We claim $\cap C_i = B_{\infty}$. Indeed, let $s \in B_{\infty}$. Then there is $i \in \mathbb{N}$ such that $j \geq i$ implies $s \in B_j$; thus for each $i \in \mathbb{N}$, $s \in C_i$, so $s \in \cap C_i$. Conversely, let $s \in S \setminus B_{\infty}$. Then there is $i \in \mathbb{N}$ such that $j \geq i$ implies $s \notin B_j$; thus $s \notin C_i$, so $s \notin \cap C_i$. Thus $B_{\infty} = \cap C_i$, so $B_{\infty} \succsim A$, as desired.

To see that lower contour sets are closed, let $A \in \mathcal{A}$ and let $(B_i) \in \mathcal{A}^{\mathbb{N}}$ be convergent such that for each $i \in \mathbb{N}$, $A \succsim B_i$. Define $B_{\infty} = \lim B_i$. Then $(S \setminus B_{\infty}) = \lim (S \setminus B_i)$. By the [Complement Lemma](#), for each $i \in \mathbb{N}$, $(S \setminus B_i) \succsim (S \setminus A)$. As upper contour sets are closed, $(S \setminus B_{\infty}) \succsim (S \setminus A)$, so by the [Complement Lemma](#), $A \succsim B_{\infty}$, as desired. ■

The [Limit-Order Lemma](#) states that for each pair of convergent sequences, if each member of the first sequence is at least the corresponding member of the second, then the limit of the first sequence is at least the limit of the second:

• **LIMIT-ORDER LEMMA:**³¹ If $(S, \mathcal{A}, \succsim)$ satisfies *order* and *continuity*, then for each pair of convergent sequences $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$ such that for each $j \in \mathbb{N}$, $A_j \succsim B_j$, we have $\lim(A_i) \succsim \lim(B_i)$.

PROOF: Let $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$ satisfy the hypothesis. Since \succsim is complete, by a standard argument³² there is $M \subseteq \mathbb{N}$ such that $(B'_i) \equiv (B_i)|_M$ is a \succsim -monotonic sequence. Define $(A'_i) \equiv (A_i)|_M$. Necessarily $\lim(A_i) = \lim(A'_i)$ and $\lim(B_i) = \lim(B'_i)$.

CASE 1: (B'_i) is non-decreasing. Then for each pair $j, k \in \mathbb{N}$ with $k \geq j$,

$$\begin{aligned} A'_k &\succsim B'_k \\ &\succsim B'_j, \end{aligned}$$

so by *continuity* $\lim(A'_i) \succsim B'_j$. Thus by *continuity*, $\lim(A'_i) \succsim \lim(B'_i)$, so $\lim(A_i) \succsim \lim(B_i)$.

³¹Variants of this result appear in [Villegas \(1964\)](#) and [Arrow \(1970\)](#). This particular result does not require \mathcal{A} to be a σ -algebra; any Hausdorff space will do, as can be seen from the proof.

³²A common proof of the Bolzano-Weierstrass Theorem includes a lemma stating that each real sequence has a monotonic subsequence; the standard proof of that lemma suffices here.

CASE 2: (B'_i) is non-increasing. Then for each pair $j, k \in \mathbb{N}$ with $k \geq j$,

$$\begin{aligned} A'_j &\succsim B'_j \\ &\succsim B'_k, \end{aligned}$$

so by *continuity* $A'_j \succsim \lim(B'_i)$. Thus by *continuity*, $\lim(A'_i) \succsim \lim(B'_i)$, so $\lim(A_i) \succsim \lim(B_i)$. ■

Appendix 2

In this appendix, we prove the [Supercabinet Blueprint Lemma](#).

• **SUPERCABINET BLUEPRINT LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a qualitative probability space that satisfies (*monotone*) *continuity* and has a supercabinet, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

PROOF: Let $\{\mathcal{Z}_v\}_{v \in \mathbb{2}} \subseteq [\mathcal{A}/\sim]$ be a supercabinet.

◦ STEP 1: For each $k \in \{0, 1, \dots\}$, there is $Z \in \mathcal{Z}_{(\frac{1}{2})^k}$ such that $Z \succ \emptyset$.

We proceed by induction on k . By [SC1], $S \in \mathcal{Z}_1$, and by *nondegeneracy* and *monotonicity*, $S \succ \emptyset$.

For the inductive hypothesis, assume $k \in \{0, 1, \dots\}$ is such that there is $Z \in \mathcal{Z}_{(\frac{1}{2})^k}$ such that $Z \succ \emptyset$. By [SC2], there are disjoint $Z', Z'' \in \mathcal{Z}_{(\frac{1}{2})^{k+1}}$ such that $Z' \cup Z'' \sim Z$. By *separability*, $Z' \succ \emptyset$, else $\emptyset \succsim Z' \cup Z'' \sim Z$, contradicting $Z \succ \emptyset$. By induction we are done. □

◦ STEP 2: For each $v \in \mathbb{2}$ such that $v > 0$, there is $Z \in \mathcal{Z}_v$ such that $Z \succ \emptyset$.

Let $v \in \mathbb{2}$ such that $v > 0$. Since $\mathbb{2}$ is dense in $[0, 1]$ there is $k \in \mathbb{2}$ such that $v > (\frac{1}{2})^k$. Since $v - (\frac{1}{2})^k \in \mathbb{2}$, by [SC2] there are disjoint $Z \in \mathcal{Z}_{(\frac{1}{2})^k}$ and $Z' \in \mathcal{Z}_{v - (\frac{1}{2})^k}$ such that $Z \cup Z' \in \mathcal{Z}_v$. By Step 1, $Z \succ \emptyset$, and by *monotonicity*, $Z' \succsim \emptyset$, so by *separability*, $Z \cup Z' \succ \emptyset$. □

◦ STEP 3: For each pair $v, v^* \in \mathbb{2}$ such that $v^* > v$, each $A \in \mathcal{Z}_{v^*}$, and each $B \in \mathcal{Z}_v$, $A \succ B$.

Let $v, v^* \in \mathbb{2}$ such that $v^* > v$, let $A \in \mathcal{Z}_{v^*}$, and let $B \in \mathcal{Z}_v$. Since $v^* - v \in \mathbb{2}$, by [SC2] there are disjoint $Z \in \mathcal{Z}_v$ and $Z' \in \mathcal{Z}_{v^* - v}$ such that $Z \cup Z' \in \mathcal{Z}_{v^*}$. Since $v^* - v > 0$ (and since $\mathcal{Z}_{v^* - v} \in [\mathcal{A}/\sim]$), by Step 2, $Z' \succ \emptyset$. Thus by *separability*, $A \sim Z \cup Z' \succ Z \sim B$. □

◦ STEP 4: Define $\mu : \mathcal{A} \rightarrow [0, 1]$.

For each $v \in \mathbb{2}$ and each $A \in \mathcal{Z}_v$, define $\mu(A) \equiv v$.

Let $A \in \mathcal{A} \setminus \bigcup_{v \in \mathbb{2}} \mathcal{Z}_v$. Define

$$\begin{aligned} \mathbb{2}^- &\equiv \{v \in \mathbb{2} \mid B \in \mathcal{Z}_v \text{ implies } A \succsim B\}, \text{ and} \\ \mathbb{2}^+ &\equiv \{v \in \mathbb{2} \mid B \in \mathcal{Z}_v \text{ implies } B \succsim A\}. \end{aligned}$$

By *monotonicity*, $S \succsim A \succsim \emptyset$, so by [SC1], $0 \in 2^-$ and $1 \in 2^+$. By Step 3, for each pair $v, v^* \in 2$ such that $v^* > v$, (1) $v^* \in 2^-$ implies $v \in 2^- \setminus 2^+$, and (2) $v \in 2^+$ implies $v^* \in 2^+ \setminus 2^-$. Thus $\inf(2^+), \sup(2^-)$ are defined, and moreover $\inf(2^+) \geq \sup(2^-)$. Since $2^- \cup 2^+ = 2$ and 2 is dense in $[0, 1]$, $\inf(2^+) = \sup(2^-)$. Define

$$\begin{aligned}\mu(A) &\equiv \inf(2^+) \\ &= \sup(2^-). \quad \square\end{aligned}$$

◦ STEP 5: For each pair $A, B \in \mathcal{A}$, $\mu(A) > \mu(B)$ implies $A \succ B$.

Let $A, B \in \mathcal{A}$ such that $\mu(A) > \mu(B)$. Since 2 is dense in $[0, 1]$, there are $v', v'' \in 2$ such that $\mu(A) > v'' > v' > \mu(B)$. Let $Z_{v'} \in \mathcal{Z}_{v'}$ and $Z_{v''} \in \mathcal{Z}_{v''}$. By Step 3, $Z_{v''} \succ Z_{v'}$.

If $A \in \cup \mathcal{Z}_v$, then by construction, $A \in \mathcal{Z}_{\mu(A)}$, so by Step 3, $A \succ Z_{v''}$. If $A \notin \cup \mathcal{Z}_v$, then by construction, $\inf\{v \in 2 \mid C \in \mathcal{Z}_v \text{ implies } C \succsim A\} > v''$, so $A \succ Z_{v''}$.

If $B \in \cup \mathcal{Z}_v$, then by construction, $B \in \mathcal{Z}_{\mu(B)}$, so by Step 3, $Z_{v'} \succ B$. If $B \notin \cup \mathcal{Z}_v$, then by construction, $v' > \sup\{v \in 2 \mid C \in \mathcal{Z}_v \text{ implies } B \succsim C\}$, so $Z_{v'} \succ B$.

Thus $A \succ Z_{v''} \succ Z_{v'} \succ B$. \square

◦ STEP 6: For each pair $A, B \in \mathcal{A}$, $\mu(A) = \mu(B)$ implies $A \sim B$.

Let $A, B \in \mathcal{A}$ such that $\mu(A) = \mu(B)$. We proceed with three cases whose proofs are similar.

CASE 1: $\mu(A) = 0$. Since 2 is dense in $[0, 1]$, there is decreasing $(v_i^+) \in 2^\mathbb{N}$ such that $\lim v_i^+ = 0$. For each $i \in \mathbb{N}$, define $v_i^- \equiv 0$. Since $\lim v_i^+ = \lim v_i^-$, by [SC3] there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^\mathbb{N}$ such that

- (i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$, and
- (ii) $\lim A_i^+ = \lim A_i^-$.

By construction, for each $i \in \mathbb{N}$, $\mu(A_i^+) > 0 = \mu(A)$, so by Step 5, $A_i^+ \succ A$. Thus by *continuity*, $\lim A_i^+ \succsim A$. By [SC1], for each $i \in \mathbb{N}$, $A_i^- \sim \emptyset$, so by *continuity*, $\lim A_i^- \sim \emptyset$. Altogether, $\emptyset \succsim A$, so by *monotonicity*, $A \sim \emptyset$. By a symmetric argument, $B \sim \emptyset$, so $A \sim B$.

CASE 2: $\mu(A) = 1$. Since 2 is dense in $[0, 1]$, there is increasing $(v_i^-) \in 2^\mathbb{N}$ such that $\lim v_i^- = 1$. For each $i \in \mathbb{N}$, define $v_i^+ \equiv 1$. Since $\lim v_i^+ = \lim v_i^-$, by [SC3] there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^\mathbb{N}$ such that

- (i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$, and
- (ii) $\lim A_i^+ = \lim A_i^-$.

By construction, for each $i \in \mathbb{N}$, $\mu(A) = 1 > \mu(A_i^-)$, so by Step 5, $A \succ A_i^-$. Thus by *continuity*, $A \succsim \lim A_i^-$. By [SC1], for each $i \in \mathbb{N}$, $A_i^+ \sim S$, so by *continuity*, $\lim A_i^+ \sim S$. Altogether, $A \succsim S$, so by *monotonicity*, $A \sim S$. By a symmetric argument, $B \sim S$, so $A \sim B$.

CASE 3: $\mu(A) \in (0, 1)$. Since 2 is dense in $[0, 1]$, there are decreasing $(v_i^+) \in 2^\mathbb{N}$ such that $\lim v_i^+ = \mu(A)$ and increasing $(v_i^-) \in 2^\mathbb{N}$ such that $\lim v_i^- = \mu(A)$. Since $\lim v_i^+ = \lim v_i^-$, by [SC3] there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^\mathbb{N}$ such that

- (i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$, and

(ii) $\lim A_i^+ = \lim A_i^-$.

By construction, for each $i \in \mathbb{N}$, $\mu(A_i^+) > \mu(A) > \mu(A_i^-)$, so by Step 5, $A_i^+ \succ A \succ A_i^-$. Thus by *continuity*, $\lim A_i^+ \succsim A \succsim \lim A_i^- = \lim A_i^+$, so $A \sim \lim A_i^+$. By a symmetric argument, $B \sim \lim A_i^+$, so $A \sim B$. \square

◦ STEP 7: μ represents \succsim .

Immediate from Step 5 and Step 6. \square

◦ STEP 8: For each disjoint pair $A, B \in \mathcal{A}$, $\mu(A) + \mu(B) \leq 1$.

Let $A, B \in \mathcal{A}$ be disjoint. Assume, by way of contradiction, $\mu(A) + \mu(B) > 1$. Since 2 is dense in $[0, 1]$, there are $v, v' \in 2$ such that $\mu(A) > v$, $\mu(B) > v'$, and $v + v' > 1$. Then $v > 1 - v'$. By [SC2], there are disjoint $Z_{v'} \in \mathcal{Z}_{v'}$ and $Z_{1-v'} \in \mathcal{Z}_{1-v'}$ such that $Z_{v'} \cup Z_{1-v'} \in \mathcal{Z}_1$. Then $Z_{v'} \cup Z_{1-v'} \sim S$. Since $\mu(A) > v > 1 - v' = \mu(Z_{1-v'})$ and $\mu(B) > v' = \mu(Z_{v'})$, by Step 7, $A \succ Z_{1-v'}$ and $B \succ Z_{v'}$. But then by the [Domination Lemma](#), $A \cup B \succ Z_{1-v'} \cup Z_{v'} \sim S$, contradicting *monotonicity*. \square

◦ STEP 9: For each disjoint pair $A, B \in \cup \mathcal{Z}_v$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Let $A, B \in \cup \mathcal{Z}_v$ be disjoint. By construction, $A \in \mathcal{Z}_{\mu(A)}$ and $B \in \mathcal{Z}_{\mu(B)}$, and by Step 8, $\mu(A) + \mu(B) \leq 1$, so by [SC2] and the [Domination Lemma](#), $A \cup B \in \mathcal{Z}_{\mu(A) + \mu(B)}$. Thus by construction, $\mu(A \cup B) = \mu(A) + \mu(B)$. \square

◦ STEP 10: For each disjoint pair $A, B \in \mathcal{A}$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Let $A, B \in \mathcal{A}$ be disjoint. By Step 8, $\mu(A) + \mu(B) \in [0, 1]$. We begin by proving that $1 > \mu(A) + \mu(B)$ implies $\mu(A) + \mu(B) \geq \mu(A \cup B)$. In this case, since 2 is dense in $[0, 1]$, there are non-increasing $(v_i), (w_i) \in 2^{\mathbb{N}}$ such that

- (1) for each $i \in \mathbb{N}$, $v_i \geq \mu(A)$, $w_i \geq \mu(B)$, and $1 \geq v_i + w_i$; and
- (2) $\lim v_i = \mu(A)$ and $\lim w_i = \mu(B)$.

By [SC4], there are convergent $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$ such that

- (1) for each $i \in \mathbb{N}$, $A_i \in \mathcal{Z}_{v_i}$ and $B_i \in \mathcal{Z}_{w_i}$; and
- (2) for each $i \in \mathbb{N}$, $A_i \cap B_i = \emptyset$.

Define $A_{\infty} \equiv \lim A_i$ and $B_{\infty} \equiv \lim B_i$. By definition of μ , for each $i \in \mathbb{N}$, $\mu(A_i) = v_i$ and $\mu(B_i) = w_i$.

Let $\epsilon > 0$. Since $\lim v_i = \mu(A)$ and $\lim w_i = \mu(B)$, there is $i^* \in \mathbb{N}$ such that $i \geq i^*$ implies $\mu(A) + \frac{\epsilon}{2} > v_i$ and $\mu(B) + \frac{\epsilon}{2} > w_i$. Then $i \geq i^*$ implies $\mu(A) + \mu(B) + \epsilon > v_i + w_i = \mu(A_i) + \mu(B_i)$, so by Step 9, $i \geq i^*$ implies $\mu(A) + \mu(B) + \epsilon > \mu(A_i \cup B_i)$.

By Step 7, $A_1 \succsim A_2 \succsim \dots$ and $B_1 \succsim B_2 \succsim \dots$. Then for each $i \in \mathbb{N}$, since A_{∞} is the limit of the subsequence $(A_j)_{j \geq i}$, thus by *continuity* $A_i \succsim A_{\infty}$, and similarly $B_i \succsim B_{\infty}$. Moreover, for each $i \in \mathbb{N}$, $A_i \cap B_i = \emptyset$; thus by the [Domination Lemma](#), $A_i \cup B_i \succsim A_{\infty} \cup B_{\infty}$, so by Step 7, $\mu(A_i \cup B_i) \geq \mu(A_{\infty} \cup B_{\infty})$.

Altogether, for each $\epsilon > 0$, there is $i \in \mathbb{N}$ such that $\mu(A) + \mu(B) + \epsilon > \mu(A_i \cup B_i) \geq \mu(A_{\infty} \cup B_{\infty})$; thus $\mu(A) + \mu(B) \geq \mu(A_{\infty} \cup B_{\infty})$.

By Step 7, for each $i \in \mathbb{N}$, $A_i \succsim A$ and $B_i \succsim B$, so by *continuity*, $A_{\infty} \succsim A$ and $B_{\infty} \succsim B$. Moreover, since for each $i \in \mathbb{N}$, $A_i \cap B_i = \emptyset$, thus $A_{\infty} \cap B_{\infty} = \emptyset$, so by

the [Domination Lemma](#), $A_\infty \cup B_\infty \succsim A \cup B$. By Step 7 and the previous observation, $\mu(A) + \mu(B) \geq \mu(A_\infty \cup B_\infty) \geq \mu(A \cup B)$, as desired.

The argument that $\mu(A) + \mu(B) > 0$ implies $\mu(A \cup B) \geq \mu(A) + \mu(B)$ is almost completely symmetric, except that applications of the [Domination Lemma](#) involve establishing that the ‘other’ pair is disjoint, which is straightforward. Altogether, then, $\mu(A \cup B) = \mu(A) + \mu(B)$ whether $\mu(A) + \mu(B)$ is 0, 1, or in between. \square

◦ STEP 11: $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

Since $\mu(S) = 1$, by Step 10 and induction, $\mu \in \mathbb{M}(\mathcal{A})$. By Step 7, μ represents \succsim , so by [Theorem V1](#), $\mu \in \mathbb{M}^\sigma(\mathcal{A})$. \square

◦ STEP 12: If $\mu' \in \mathbb{M}^\sigma(\mathcal{A})$ represents \succsim , then $\mu' = \mu$.

If $\mu' \in \mathbb{M}^\sigma(\mathcal{A})$ represents \succsim , then by [SC1] and [SC2], it is immediate that μ' must be defined as in Step 4. \blacksquare

Appendix 3

In this appendix, we prove our primary lemmas about greedy transforms: the [Idempotence Lemma](#), the [1-AS Lemma](#), the [Greedy Removal Lemma](#), and the [Convergence Lemma](#). We abuse language in our informal summaries of these lemmas, writing that an event satisfies k -AS instead of writing that its associated subspace does.

The [Idempotence Lemma](#) states that each greedy transform is idempotent, and for convenience includes the easy corollary that two images of a greedy transform that are equally likely are in fact equivalent.

• **IDEMPOTENCE LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *(monotone) continuity*, then for each pair $A, B \in \mathcal{A}$, $\mathcal{G}^A(\mathcal{G}^A(B)) = \mathcal{G}^A(B)$. Moreover, for each $A \in \mathcal{A}$ and each pair $B, B' \in \mathcal{A}$, $\mathcal{G}^A(B) \sim \mathcal{G}^A(B')$ implies $\mathcal{G}^A(B) = \mathcal{G}^A(B')$.

PROOF: We proceed by induction on s , covering the base case with our inductive hypothesis. Assume $s \in S$ is such that for each $s' \in S$,

$$\mathcal{G}^A(\mathcal{G}^A(B)) \cap \{s' \in S | s' < s\} = \mathcal{G}^A(B) \cap \{s' \in S | s' < s\}.$$

If $s \in \mathcal{G}^A(B)$, then by construction,

$$\begin{aligned} \mathcal{G}^A(B) &\succsim [\mathcal{G}^A(B) \cap \{s' \in S | s' < s\}] \cup \{s\} \\ &= [\mathcal{G}^A(\mathcal{G}^A(B)) \cap \{s' \in S | s' < s\}] \cup \{s\}, \end{aligned}$$

so by construction, $s \in \mathcal{G}^A(\mathcal{G}^A(B))$.

If $s \notin \mathcal{G}^A(B)$, then by construction,

$$\begin{aligned} [\mathcal{G}^A(\mathcal{G}^A(B)) \cap \{s' \in S | s' < s\}] \cup \{s\} &= [\mathcal{G}^A(B) \cap \{s' \in S | s' < s\}] \cup \{s\} \\ &\succ B \\ &\succsim \mathcal{G}^A(B), \end{aligned}$$

so by construction, $s \notin \mathcal{G}^A(\mathcal{G}^A(B))$. Thus

$$\mathcal{G}^A(\mathcal{G}^A(B)) \cap \{s' \in S \mid s' < s + 1\} = \mathcal{G}^A(B) \cap \{s' \in S \mid s' < s + 1\}.$$

By induction, $\mathcal{G}^A(\mathcal{G}^A(B)) = \mathcal{G}^A(B)$.

Now assume $A \in \mathcal{A}$ and $B, B' \in \mathcal{A}$ are such that $\mathcal{G}^A(B) \sim \mathcal{G}^A(B')$. Then $\mathcal{G}^A(B) = \mathcal{G}^A(\mathcal{G}^A(B)) = \mathcal{G}^A(\mathcal{G}^A(B')) = \mathcal{G}^A(B')$, as desired. ■

The [1-AS Lemma](#) is the ordinal analogue of Kakeya's observation ([Kakeya, 1914](#); [Kakeya, 1915](#)): 1-AS of A guarantees that applying A 's greedy transform to an event B that is no larger than A yields an event as likely as B .

• **1-AS LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *(monotone) continuity*, then for each $A \in \mathcal{A}$ such that $\succsim|_A$ satisfies 1-AS and each $B \in \mathcal{A}$ such that $A \succsim B$, $\mathcal{G}^A(B) \sim B$.

PROOF: If $A = \emptyset$ the result is trivial, so assume $A \neq \emptyset$. Then by 1-AS, $|A| = |\mathbb{N}|$. For convenience, re-index S so that (1) $A = \mathbb{N}$, (2) index order is preserved for A , and (3) members of $S \setminus A$ are not indexed by natural numbers. Since $B \succsim \mathcal{G}^A(B)$, it suffices to show $\mathcal{G}^A(B) \succsim B$.

CASE 1: $|A \setminus \mathcal{G}^A(B)| = 0$. Then $\mathcal{G}^A(B) = A \succsim B$.

CASE 2: $0 < |A \setminus \mathcal{G}^A(B)| < |\mathbb{N}|$. Define $s^* \equiv \max(A \setminus \mathcal{G}^A(B))$. By construction, $[\mathcal{G}^A(B) \setminus \{s^* + 1, s^* + 2, \dots\} \cup \{s^*\}] \succ B$. Since $\succsim|_A$ satisfies 1-AS, thus by *separability*

$$\begin{aligned} \mathcal{G}^A(B) &= [\mathcal{G}^A(B) \setminus \{s^* + 1, s^* + 2, \dots\}] \cup \{s^* + 1, s^* + 2, \dots\} \\ &\succsim [\mathcal{G}^A(B) \setminus \{s^* + 1, s^* + 2, \dots\}] \cup \{s^*\} \\ &\succ B. \end{aligned}$$

But this contradicts $B \succsim \mathcal{G}^A(B)$, so in fact Case 2 is impossible.

CASE 3: $|A \setminus \mathcal{G}^A(B)| = |\mathbb{N}|$. By construction, for each $s \in A \setminus \mathcal{G}^A(B)$, $[\mathcal{G}^A(B) \setminus \{s + 1, s + 2, \dots\}] \cup \{s\} \succ B$. Thus by *continuity*,

$$\begin{aligned} \mathcal{G}^A(B) &= \lim_{s \in S \setminus \mathcal{G}^A(B)} [\mathcal{G}^A(B) \setminus \{s + 1, s + 2, \dots\}] \cup \{s\} \\ &\succsim B. \quad \blacksquare \end{aligned}$$

The [Greedy Removal Lemma](#) is the ordinal analogue of Komornik's observation ([Komornik, 2015](#)): if A is $(k + 1)$ -AS, then removing an image of its greedy transform yields a subevent that is k -AS.

• **GREEDY REMOVAL LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *(monotone) continuity*, then for each $k \in \mathbb{N}$, each $A \in \mathcal{A}$ such that $\succsim|_A$ satisfies $(k + 1)$ -AS, and each $B \in \mathcal{A}$,

$$\succsim|_{A \setminus \mathcal{G}^A(B)} \text{ satisfies } k\text{-AS.}$$

PROOF: Let $k \in \mathbb{N}$ and assume $\succsim|_A$ satisfies k -AS. If $A = \emptyset$ or $B \succsim A$, the result is trivial, so assume $A \succ B \succsim \emptyset$. By 1-AS, $|A| = |\mathbb{N}|$. For convenience, re-index S so that (i) $A = \mathbb{N}$, (2) index order is preserved for A , and (iii) members of $S \setminus A$ are not indexed by natural numbers.

◦ STEP 1: If $k' < k + 1$ greedy transform sets G_i are iteratively removed from A , and if $A \setminus \cup G_i$ is nonempty, then $\succsim|_{A \setminus \cup G_i}$ satisfies 1-AS.

Let $k' \in \mathbb{N}$ such that $k' < k + 1$, and let $B_1, \dots, B_{k'} \in \mathcal{A}$. Define $A_1 \equiv A$, and for each $i \in \{1, 2, \dots, k'\}$, define:

- $G_i \equiv \mathcal{G}^{A_i}(B_i)$,
- $A_{i+1} \equiv A_i \setminus G_i$.

By construction, the G_i are pairwise disjoint. We claim $A \setminus \cup G_i \neq \emptyset$ implies $\succsim|_{A \setminus \cup G_i}$ satisfies 1-AS. Indeed, let $s \in A \setminus \cup G_i$. Since $\succsim|_A$ satisfies $(k+1)$ -AS, there are pairwise disjoint $B_1, \dots, B_{k+1} \subseteq \{s+1, s+2, \dots\}$ such that for each $i \in \{1, 2, \dots, k+1\}$, $B_i \succsim \{s\}$. By construction, for each $i \in \{1, 2, \dots, k'\}$, $\{s\} \succsim G_i \cap \{s+1, s+2, \dots\}$. Thus for each $i \in \{1, 2, \dots, k'\}$, $B_i \succsim G_i \cap \{s+1, s+2, \dots\}$. By repeated application of the [Domination Lemma](#),

$$\begin{aligned} B_1 &\succsim G_1 \cap \{s+1, s+2, \dots\}, \\ B_1 \cup B_2 &\succsim (G_1 \cap \{s+1, s+2, \dots\}) \cup (G_2 \cap \{s+1, s+2, \dots\}), \\ &\vdots \\ \bigcup_{i=1}^{k'} B_i &\succsim \bigcup_{i=1}^{k'} (G_i \cap \{s+1, s+2, \dots\}) \end{aligned}$$

Thus if $\{s\} \succ [\{s+1, s+2, \dots\} \setminus \cup G_i]$, then $B_{k'+1} \succ [\{s+1, s+2, \dots\} \setminus \cup G_i]$, so by the [Domination Lemma](#),

$$\begin{aligned} \bigcup_{i=1}^{k'+1} B_i &\succ \bigcup_{i=1}^{k'} (G_i \cap [A \cap \{s+1, s+2, \dots\}]) \cup [\{s+1, s+2, \dots\} \setminus \cup G_i] \\ &= \{s+1, s+2, \dots\}, \end{aligned}$$

contradicting *monotonicity*. Thus $[\{s+1, s+2, \dots\} \setminus \cup G_i] \succsim \{s\}$. Since $s \in A \setminus G_i$ was arbitrary, thus $\succsim|_{A \setminus \cup G_i}$ satisfies 1-AS. \square

◦ STEP 2: Conclude.

Let $B \in \mathcal{A}$ such that $A \succ B$ and let $G_1 = \mathcal{G}^A(B)$. Then there is $s \in A \setminus G_1$, else by *monotonicity* $B \succsim G_1 = A \succ B$, contradicting $B \sim B$. By Step 1, $\succsim|_{A \setminus G_1}$ satisfies 1-AS.

Let $s \in A \setminus G_1$ and define $A_1 \equiv \{s+1, s+2, \dots\}$. Then $\succsim|_{A_1}$ satisfies $(k+1)$ -AS and $\succsim|_{A_1 \setminus G_1}$ satisfies 1-AS. Thus there are pairwise disjoint $B_1, \dots, B_{k+1} \subseteq A_1$ such that for each $i \in \{1, 2, \dots, k+1\}$, $B_i \succsim \{s\}$.

Define $A_2 \equiv A_1 \setminus G_1$, and for each $i \in \{2, 3, \dots, k+1\}$, define:

- $G_i \equiv \mathcal{G}^{\leq}(A_i, \{s\})$,

- $A_{i+1} \equiv A_i \setminus G_i$.

By construction, the G_i are pairwise disjoint. Assume, by way of contradiction, there is $i \in \{2, 3, \dots, k+1\}$ such that $\{s\} \succ A_i$. Let i^* be the least such i . Then $B_{i^*} \succsim \{s\} \succ A_{i^*}$, and for each $i \in \{1, \dots, i^* - 1\}$, $B_i \succsim \{s\} \succsim G_i$. But then by repeated application of the [Domination Lemma](#) as in Step 1,

$$\begin{aligned} \bigcup_{i=1}^{i^*-1} B_i &\succ \bigcup_{i=1}^{i^*-1} G_i \cup A_{i^*} \\ &= A_1, \end{aligned}$$

contradicting *monotonicity*.

Thus for each $i \in \{2, 3, \dots, k+1\}$, $A_i \succsim \{s\}$, so by *monotonicity* A_i is nonempty, so by Step 1 $\succsim|_{A_i}$ satisfies 1-AS, so by the 1-AS Lemma $B_i \sim \{s\}$. Since $s \in A \setminus G_1$ was arbitrary, $\succsim|_{A \setminus G_1}$ satisfies k -AS. ■

The [Convergence Lemma](#) concerns generalized greedy transforms, each of which takes as input a list of events and outputs the same number of events:

DEFINITION: For each $A \in \mathcal{A}$ and each $k \in \mathbb{N}$, the *greedy transform* $\mathcal{G}^A : \mathcal{A}^k \rightarrow \mathcal{A}^k$ is defined as follows. For each $(B_1, B_2, \dots, B_k) \in \mathcal{A}^k$, $\mathcal{G}^A(B_1, B_2, \dots, B_k) \in \mathcal{A}^k$ is defined recursively by:

- $\mathcal{G}_0^A(B_1, B_2, \dots, B_k) \equiv \emptyset$, and
- for each $i \in \{1, 2, \dots, k\}$,

$$\mathcal{G}_i^A(B_1, B_2, \dots, B_k) \equiv \mathcal{G}^{S \setminus (\cup_{j < i} \mathcal{G}_j^A(B_1, B_2, \dots, B_k))}(A_i).$$

The lemma states that for nice sequences of event lists, the associated sequence of event lists output by a generalized greedy transform converges.

• **CONVERGENCE LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying (*monotone*) *continuity*, $k \in \mathbb{N}$, and $(B_i^1), (B_i^2), \dots, (B_i^k) \in \mathcal{A}^\mathbb{N}$ are such that

- (1) for each $j \in \{1, 2, \dots, k\}$, (B_i^j) is monotonic, and
- (2) for each $i \in \mathbb{N}$, $B_i^1, B_i^2, \dots, B_i^k$ are pairwise disjoint,

then for each $A \in \mathcal{A}$ and each $j \in \{1, 2, \dots, k\}$, $(\mathcal{G}_j^A(B_i^1, B_i^2, \dots, B_i^k)) \in \mathcal{A}^\mathbb{N}$ is convergent.

PROOF: Let $A \in \mathcal{A}$, let $k \in \mathbb{N}$, and let $(B_i^1), (B_i^2), \dots, (B_i^k) \in \mathcal{A}^\mathbb{N}$ satisfy the hypotheses. For each $i \in \mathbb{N}$ and each $j \in \{1, 2, \dots, k\}$, define $G_i^j \in \mathcal{A}$ by:

$$G_i^j \equiv \mathcal{G}_j^A(B_i^1, B_i^2, \dots, B_i^k).$$

To prove $(G_i^1), (G_i^2), \dots, (G_i^k)$ are convergent, we proceed by induction. We cover the base step with our inductive hypothesis on j^* : assume $j^* \in \{1, 2, \dots, k-1\}$ is such that for each $j \in \{1, 2, \dots, k\}$ such that $j < j^*$, (G_i^j) is convergent. We claim $(G_i^{j^*})$ is convergent.

Within the current inductive argument, we make a second inductive argument. We cover the base step with our inductive hypothesis on s : assume $s \in S$ is such that for each $s' < s$, there is $i^* \in \mathbb{N}$ such that for each $j \in \{1, 2, \dots, j^*\}$, either

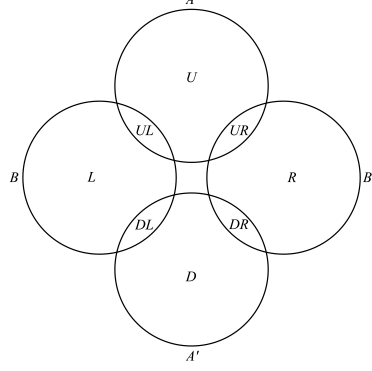


Figure 4: Euler diagram for A, B, A', B' . For example, $UL \equiv A \cap B$ and $D \equiv A' \setminus (B \cup B')$.

- (1) $i \geq i^*$ implies $s' \in G_i^j$, or
- (2) $i \geq i^*$ implies $s' \in G_i^j$.

By the inductive hypothesis on j^* , there is $i^* \in \mathbb{N}$ such that for each $j \in \{1, 2, \dots, k\}$ such that $j < j^*$, either

- (1) $i \geq i^*$ implies $s \in G_i^j$, or
- (2) $i \geq i^*$ implies $s \in G_i^j$.

Assume, by way of contradiction, that for each $i^{**} \in \mathbb{N}$ there are $i_1 \geq i^{**}$ and $i_2 \geq i^{**}$ such that $s \in G_{i_1}^{j^*}$ and $s \notin G_{i_2}^{j^*+1}$. Then there are $i_1, i_2, i_3 \in \mathbb{N}$ with $i_3 > i_2 > i_1 > i^*$ such that $s \in G_{i_1}^{j^*}$, $s \notin G_{i_2}^{j^*}$, and $s \in G_{i_3}^{j^*}$. By definition of i^* , $G_{i_1}^{j^*} \cap \{s' \in S \mid s' < s\} = G_{i_2}^{j^*} \cap \{s' \in S \mid s' < s\} = G_{i_3}^{j^*} \cap \{s' \in S \mid s' < s\}$. Thus by construction, $B_{i_1}^{j^*} \succeq G_{i_1}^{j^*} \cap \{s' \in S \mid s' < s\} \cup \{s\} \succ B_{i_2}^{j^*}$, so $(B_i^{j^*})$ is non-increasing. But then $B_{i_3}^{j^*} \succeq G_{i_1}^{j^*} \cap \{s' \in S \mid s' < s\} \cup \{s\} \succ B_{i_2}^{j^*}$, contradicting that $(B_i^{j^*})$ is non-increasing.

By induction on s , $(G_i^{j^*})$ is convergent. By induction on j^* , $(G_i^1), (G_i^2), \dots, (G_i^k)$ are convergent. ■

Appendix 4

In this appendix, we prove our primary lemmas about qualitative halves: the [Half-Equivalence Lemma](#), the [First Halving Lemma](#), and the [Second Halving Lemma](#).

The [Half-Equivalence Lemma](#) states that for any two disjoint pairs whose unions are equally likely such that each pair's members are equally likely, all four events are equally likely.

• **HALF-EQUIVALENCE LEMMA:** If $(S, \mathcal{A}, \succeq)$ is a qualitative probability space, then for each disjoint pair $A, A' \in \mathcal{A}$ and each disjoint pair $B, B' \in \mathcal{A}$, if

- (1) $A \sim A'$,
- (2) $B \sim B'$, and
- (3) $A \cup A' \sim B \cup B'$,

then $A \sim B$.

PROOF: For convenience, label the components of the Euler diagram for A, A', B, B' according to [Figure 4](#):

Assume, by way of contradiction, $A \not\sim B$; without loss of generality, assume $A \succ B$.

We claim $DL \cup D \succ UR \cup R$. Otherwise, by *separability*,

$$\begin{aligned} B &\sim B' \\ &= DR \cup (UR \cup R) \\ &\succsim DR \cup (DL \cup D) \\ &= A' \\ &\sim A, \end{aligned}$$

contradicting $A \succ B$.

We claim $L \cup R \succsim U \cup D$. Otherwise, by *separability*,

$$\begin{aligned} A \cup A' &= (UL \cup UR \cup DL \cup DR) \cup (U \cup D) \\ &\succ (UL \cup UR \cup DL \cup DR) \cup (L \cup R) \\ &= B \cup B', \end{aligned}$$

contradicting $A \cup A' \sim B \cup B'$. Similarly, $U \cup D \succsim L \cup R$, so $L \cup R \sim U \cup D$.

But then by *separability*,

$$\begin{aligned} (L \cup R) \cup (UL \cup UR) &\sim (U \cup D) \cup (UL \cup UR) \\ &= A \cup D \\ &\succ B \cup D \\ &= (UL \cup L) \cup (DL \cup D) \\ &\succ (UL \cup L) \cup (UR \cup R), \end{aligned}$$

contradicting $L \cup R \cup UL \cup UR \sim L \cup R \cup UL \cup UR$. ■

The **First Halving Lemma** states any event satisfying 1- AS can be associated with two disjoint subevents analogous to its halves:

• **FIRST HALVING LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying (*monotone*) *continuity*, then for each $A \in \mathcal{A}$ such that $A \succ \emptyset$ and $\succsim|_A$ satisfies 1- AS , there is $H \subseteq A$ such that $H \sim A \setminus H$ and $A \succ H \succ \emptyset$.

PROOF: Since $A \neq \emptyset$, by 1- AS , $|A| = |\mathbb{N}|$. Assume, without loss of generality, $A = S = \mathbb{N}$. Let $\mathbb{C} \subseteq [0, 1]$ be the canonical Cantor set (Cantor, 1883). Define $\Psi : \mathcal{A} \rightarrow \mathbb{C}$ by

$$\Psi(B) = \sum_{s \in B} \frac{2^s}{3^s}.$$

It is well-known that Ψ is an order-preserving homeomorphism when \mathcal{A} has the lexicographic order $>_{\text{LEX}}$ and \mathbb{C} has the usual order $>$; thus each closed collection $\mathcal{A}' \subseteq \mathcal{A}$ contains its $>_{\text{LEX}}$ -supremum. Furthermore, for each $B \in \mathcal{A}$, $\Psi(B^c) = 1 - \Psi(B)$.

Define the collection of events $\mathcal{A}^- \subseteq \mathcal{A}$ by:

$$\mathcal{A}^- \equiv \{B \in \mathcal{A} \mid B^c \succsim B\}$$

We claim \mathcal{A}^- is closed. Indeed, let $(B_i) \in (\mathcal{A}^-)^\mathbb{N}$ be convergent and define $B \equiv \lim(B_i)$. Then

$$\begin{aligned} \lim(B_i^c) &= \lim(\Psi^{-1}(\Psi(B_i^c))) \\ &= \lim(\Psi^{-1}(1 - \Psi(B_i))) \\ &= \Psi^{-1}(\lim(1 - \Psi(B_i))) \\ &= \Psi^{-1}(1 - \lim \Psi(B_i)) \\ &= \Psi^{-1}(1 - \Psi(B)) \\ &= B^c. \end{aligned}$$

Thus $(B_i), (B_i^c)$ are convergent such that for each $i \in \mathbb{N}$, $B_i^c \succsim B_i$, so by the [Limit-Order Lemma](#),

$$\begin{aligned} \lim(B)^c &= B^c \\ &= \lim(B_i^c) \\ &\succsim \lim(B_i), \end{aligned}$$

and hence $\lim(B_i) \in \mathcal{A}^-$. Thus \mathcal{A}^- is closed, so it contains its $>_{\text{LEX}}$ -supremum H .

Since $H = S$ implies $H \succ H^c$, contradicting $H \in \mathcal{A}^-$, thus $H^c \neq \emptyset$. Similarly, $H^c \neq S$. Assume, by way of contradiction, $|H^c| < |\mathbb{N}|$. Define $s^* \equiv \max H^c$. By 1-AS and *separability*,

$$\begin{aligned} H &= [H \setminus \{s^* + 1, s^* + 2, \dots\}] \cup \{s^* + 1, s^* + 2, \dots\} \\ &\succsim [H \setminus \{s^* + 1, s^* + 2, \dots\}] \cup \{s^*\}. \end{aligned}$$

Then by the [Complement Lemma](#) and *separability*,

$$\begin{aligned} ([H \setminus \{s^* + 1, s^* + 2, \dots\}] \cup \{s^*\})^c &\succsim H^c \\ &\succsim H \\ &\succsim [H \setminus \{s^* + 1, s^* + 2, \dots\}] \cup \{s^*\}. \end{aligned}$$

But then $[H \setminus \{s^* + 1, s^* + 2, \dots\}] \cup \{s^*\} \in \mathcal{A}^-$, contradicting the $>_{\text{LEX}}$ -maximality of H . Thus $|H^c| = |\mathbb{N}|$.

Finally, for each $s \in H^c$, $H \cup \{s\} \notin \mathcal{A}^-$, else $H \cup \{s\}$ would contradict the $>_{\text{LEX}}$ -maximality of H in \mathcal{A}^- . Thus by the [Limit-Order Lemma](#),

$$\begin{aligned} H &= \lim_{s \in H^c} (H \cup \{s\}) \\ &\succsim \lim_{s \in H^c} (H \cup \{s\})^c \\ &= H^c, \end{aligned}$$

and thus $H \sim H^c$.

Since $A \succ \emptyset$, necessarily $A \succ H$, else $H \succsim A$ and $H^c \sim H \succsim A \succ \emptyset$, so by the [Domination Lemma](#), $A \sim H \cup H^c \succ A$, contradicting $A \sim A$. Necessarily $H \succ \emptyset$, else $\emptyset \succsim H$ and $\emptyset \succsim H \sim H^c$, so by the [Domination Lemma](#), $\emptyset \succsim H \cup H^c \sim A$, contradicting $A \succ \emptyset$. Thus $A \succ H \succ \emptyset$. ■

The [Second Halving Lemma](#) states that under 2-AS, each event can be associated with two disjoint events analogous to its halves:

• **SECOND HALVING LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *(monotone) continuity* and 2-AS, then for each $A \in \mathcal{A}$ such that $A \succ \emptyset$, there are disjoint $H(A), H'(A) \in \mathcal{A}$ such that

- (1) $A \sim H(A) \cup H'(A)$, and
- (2) $A \succ H(A) \sim H'(A) \succ \emptyset$.

PROOF: Let $A \in \mathcal{A}$ such that $A \succ \emptyset$. Since \succsim satisfies 1-AS and, by *monotonicity*, $S \succsim S \setminus A$, thus by the [1-AS Lemma](#), $\mathcal{G}^S(S \setminus A) \sim S \setminus A$. By the [Complement Lemma](#),

- (1) $S \succ S \setminus A \sim \mathcal{G}^S(S \setminus A)$, and
- (2) $S \setminus \mathcal{G}^S(S \setminus A) \sim S \setminus (S \setminus A) = A \succ \emptyset$.

Since \succsim satisfies 2-AS, by the [Greedy Removal Lemma](#), $\succsim|_{S \setminus \mathcal{G}^S(S \setminus A)}$ satisfies 1-AS. By the [First Halving Lemma](#), there are disjoint $H(A), H'(A) \subseteq S \setminus \mathcal{G}^S(S \setminus A)$ such that

- (1) $A \sim S \setminus \mathcal{G}^S(S \setminus A) = H(A) \cup H'(A)$, and
- (2) $A \sim S \setminus \mathcal{G}^S(S \setminus A) \succ H(A) \sim H'(A) \succ \emptyset$,

as desired. ■

Appendix 5

In this appendix, we prove the [Supercabinet Construction Lemma](#) and [Theorem 1](#).

• **SUPERCABINET CONSTRUCTION LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *(monotone) continuity* and 3-AS, then it has a supercabinet.

PROOF: The only notation carried from one step to the next is the notation in the step's statement.

◦ STEP 1: Define $\{A_q^1\}_{q \in \{0,1,\dots\}}, \{H(A_q^1)\}_{q \in \{0,1,\dots\}}, \{H'(A_q^1)\}_{q \in \{0,1,\dots\}} \subseteq \mathcal{A}$ such that for each $q \in \{0,1,\dots\}$,

- (1) $H(A_q^1) \cap H'(A_q^1) = \emptyset$,
- (2) $A_q^1 \sim H(A_q^1) \cup H'(A_q^1)$, and
- (3) $A_q^1 \succ A_{q+1}^1 \sim H(A_q^1) \sim H'(A_q^1) \succ \emptyset$.

We proceed recursively. Define $A_0^1 \equiv \mathcal{G}^S(S) = S$; by *monotonicity* and *nondegeneracy*, $A_0^1 \succ \emptyset$.

Suppose we have $A_q^1 \in \mathcal{A}$ such that $A_q^1 \succ \emptyset$. Since \succsim satisfies 2-AS, by the [Second Halving Lemma](#), there are disjoint $H(A_q^1), H'(A_q^1)$ such that

- (1) $A_q^1 \sim H(A_q^1) \cup H'(A_q^1)$, and
- (2) $A_q^1 \succ H(A_q^1) \sim H'(A_q^1) \succ \emptyset$.

Define $A_{q+1}^1 \equiv \mathcal{G}^S(H(A_q^1))$; by the [Half-Equivalence Lemma](#) this is well-defined. Since \succsim satisfies 1-AS and, by *monotonicity*, $S \succsim H(A_q^1)$, thus by the [1-AS Lemma](#), $A_{q+1}^1 \sim H(A_q^1)$. □

◦ STEP 2: For each $q \in \{0,1,\dots\}$, define $\{A_q^p\}_{p \in \{0,1,\dots,2^q\}} \subseteq \mathcal{A}$ such that for each $p \in \{0,1,\dots,2^q-1\}$,

$$A_q^{p+1} \sim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1).$$

We proceed recursively. For each $q \in \mathbb{N}$, define $A_q^0 \equiv \emptyset$. For each $q \in \{0, 1, \dots\}$ and each $p \in \{0, 1, \dots, 2^q - 1\}$, define

$$A_q^{p+1} \equiv \mathcal{G}^S(A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1)).$$

Since \succsim satisfies 1-AS and, by *monotonicity*, $S \succsim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1)$, thus by the [1-AS Lemma](#), $A_q^{p+1} \sim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1)$.

By the [Idempotence Lemma](#), this definition gives the same $\{A_q^1\}_{q \in \{0, 1, \dots\}}$ defined before. \square

◦ STEP 3: For each $q \in \{0, 1, \dots\}$ and each $p \in \{0, 1, \dots, 2^q\}$, $\succsim|_{S \setminus A_q^p}$ satisfies 2-AS.

Let $q \in \{0, 1, \dots\}$ and $p \in \{0, 1, \dots, 2^q\}$ be such that $A_q^p \neq S$. Since \succsim satisfies 3-AS, thus by the [Greedy Removal Lemma](#) and the [Idempotence Lemma](#), $\succsim|_{S \setminus A_q^p} = \succsim|_{S \setminus \mathcal{G}^S(A_q^p)}$ satisfies 2-AS. \square

◦ STEP 4: For each $q \in \{0, 1, \dots\}$,

- (1) $p \in \{0, 1, \dots, 2^q - 1\}$ implies $\mathcal{G}^{S \setminus A_q^p}(A_q^1) \sim A_q^1$, and
- (2) $p \in \{0, 1, \dots, 2^q\}$ implies $A_q^p = A_{q+1}^{2p}$.

We proceed by induction on q . For the base step, let $q = 0$. Then $\mathcal{G}^{S \setminus A_q^0}(A_q^1) = \mathcal{G}^S(S) = S = A_q^1$. For the inductive hypothesis, assume $q \in \{0, 1, \dots\}$ is such that for each $p \in \{0, 1, \dots, 2^q - 1\}$, $\mathcal{G}^{S \setminus A_q^p}(A_q^1) \sim A_q^1$.

Within the current inductive argument, we make a second inductive argument, on p . For the base step, $A_q^0 = \emptyset = A_{q+1}^0$. For the inductive hypothesis, assume $p \in \{0, 1, \dots, 2^q - 1\}$ is such that $A_q^p = A_{q+1}^{2p}$. For convenience, define $G_1, G_2, G'_2 \in \mathcal{A}$ by

- $G_1 \equiv \mathcal{G}^{S \setminus A_{q+1}^{2p}}(A_{q+1}^1)$,
- $G_2 \equiv \mathcal{G}^{S \setminus A_{q+1}^{2p+1}}(A_{q+1}^1)$, and
- $G'_2 \equiv \mathcal{G}^{S \setminus [A_{q+1}^{2p} \cup G_1]}(A_{q+1}^1)$.

We make three claims, which we prove in sequence:

CLAIM 1: $G_1 \sim A_{q+1}^1$,

CLAIM 2: $G_2 \sim A_{q+1}^1$, and

CLAIM 3: $A_q^{p+1} = A_{q+1}^{2(p+1)}$.

PROOF OF CLAIM 1: By the inductive hypothesis on p , $A_q^p = A_{q+1}^{2p}$, so by Step 1, *monotonicity*, and the inductive hypothesis on q ,

$$\begin{aligned} S \setminus A_{q+1}^{2p} &= S \setminus A_q^p \\ &\succsim \mathcal{G}^{S \setminus A_q^p}(A_q^1) \\ &\sim A_q^1 \\ &\succsim A_{q+1}^1. \end{aligned}$$

Since by Step 3, $\lesssim|_{S \setminus A_{q+1}^{2p}} = \lesssim|_{S \setminus A^p}$ satisfies 2-AS, thus by the [1-AS Lemma](#), $G_1 = \mathcal{G}^{S \setminus A_{q+1}^{2p}}(A_{q+1}^1) \sim A_{q+1}^1$.

PROOF OF CLAIM 2: Since by Claim 1, $H(A_q^1) \sim A_{q+1}^1 \sim G_1$, necessarily $S \setminus [A_{q+1}^{2p} \cup G_1] \lesssim H'(A_q^1)$, else by the [Domination Lemma](#), the hypothesis on p , and the hypothesis on q ,

$$\begin{aligned} A_q^1 &\sim H(A_q^1) \cup H'(A_q^1) \\ &\succ G_1 \cup S \setminus [A_{q+1}^{2p} \cup G_1] \\ &= S \setminus A_{q+1}^{2p} \\ &= S \setminus A_q^p \\ &\lesssim \mathcal{G}^{S \setminus A_q^p}(A_q^1) \\ &\sim A_q^1, \end{aligned}$$

contradicting $A_q^1 \sim A_{q+1}^1$.

By Step 2, $A_{q+1}^{2p+1} \sim A_{q+1}^{2p} \cup G_1$. By the [Complement Lemma](#), $S \setminus A_{q+1}^{2p+1} \sim S \setminus [A_{q+1}^{2p} \cup G_1] \lesssim H'(A_q^1) \sim A_{q+1}^1$. By Step 3, $\lesssim|_{S \setminus A_{q+1}^{2p+1}}$ satisfies 2-AS, so by the [1-AS Lemma](#), $G_2 = \mathcal{G}^{S \setminus A_{q+1}^{2p+1}}(A_{q+1}^1) \sim A_{q+1}^1$.

PROOF OF CLAIM 3: As argued in the proof of Claim 2, $S \setminus [A_{q+1}^{2p} \cup G_1] \lesssim H'(A_q^1) \sim A_{q+1}^1$. By Step 3, $\lesssim|_{S \setminus A_{q+1}^{2p}}$ satisfies 2-AS, so by the [Greedy Removal Lemma](#), $\lesssim|_{(S \setminus A_{q+1}^{2p}) \setminus G_1} = \lesssim|_{S \setminus [A_{q+1}^{2p} \cup G_1]}$ satisfies 1-AS. Thus by the [1-AS Lemma](#), $G'_2 \sim A_{q+1}^1$.

Since $H(A_q^1) \sim G_1$ and $H'(A_q^1) \sim G'_2$, thus by the [Domination Lemma](#), $A_q^1 \sim H(A_q^1) \cup H'(A_q^1) \sim G_1 \cup G'_2$. By the hypothesis on q , $\mathcal{G}^{S \setminus A_q^p}(A_q^1) \sim A_q^1 \sim G_1 \cup G'_2$. By Step 2, the hypothesis on p , and *separability*,

$$\begin{aligned} A_q^{p+1} &\sim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1) \\ &= A_{q+1}^{2p} \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1) \\ &\sim A_{q+1}^{2p} \cup (G_1 \cup G'_2) \\ &= [A_{q+1}^{2p} \cup G_1] \cup G'_2 \end{aligned}$$

By Step 2, $A_{q+1}^{2p} \cup G_1 \sim A_{q+1}^{2p+1}$. By the first paragraph of this claim's proof and Claim 2, $G'_2 \sim A_{q+1}^1 \sim G_2$. Thus by the [Domination Lemma](#), $[A_{q+1}^{2p} \cup G_1] \cup G'_2 \sim A_{q+1}^{2p+1} \cup G_2$. By Step 2, $A_{q+1}^{2p+1} \cup G_2 \sim A_{q+1}^{2p+2}$. Altogether, $A_q^{p+1} \sim A_{q+1}^{2p+2}$.

By induction on p , we conclude the following: if $q \in \{0, 1, \dots\}$ is such that for each $p \in \{0, 1, \dots, 2^q - 1\}$, $\mathcal{G}^{S \setminus A_q^p}(A_q^1) \sim A_q^1$, then

- (1) for each $p \in \{0, 1, \dots, 2^{q+1} - 1\}$, $\mathcal{G}^{S \setminus A_{q+1}^p}(A_{q+1}^1) \sim A_{q+1}^1$, and
- (2) for each $p \in \{0, 1, \dots, 2^q\}$, $A_q^p = A_{q+1}^{2p}$.

By induction on q , we are done. \square

◦ STEP 5: For each $q \in \{0, 1, \dots\}$ and each $p \in \{0, 1, \dots, 2^q\}$, $A_q^p \sim S \setminus A_q^{2^q-p}$.

Let $q \in \{0, 1, \dots\}$. For each $p \in \{0, 1, \dots, 2^q\}$, define $B_q^p \equiv S \setminus A_q^{2^q-p}$. We proceed by induction on p .

For the base step, let $p = 0$. By Step 4, $B_q^p = S \setminus A_q^{2^q} = S \setminus A_0^1 = S \setminus S = \emptyset$, so $A_q^0 \sim B_q^0$. For the inductive hypothesis, assume $p \in \{0, 1, \dots, 2^q - 1\}$ is such that $A_q^p \sim B_q^p$.

Define $A, B, C', B', A' \in \mathcal{A}$ by:

$$\begin{aligned} A &\equiv A_q^p, \\ B &\equiv \mathcal{G}^{S \setminus A}(A_q^1), \\ C' &\equiv A_q^{2^q - (p+1)} \\ B' &\equiv \mathcal{G}^{S \setminus C'}(A_q^1) \\ A' &\equiv S \setminus (C' \cup B') \end{aligned}$$

By Step 2, $C' \cup B' \sim A_q^{2^q - p}$. By the [Complement Lemma](#), $A' = S \setminus (C' \cup B') \sim S \setminus A_q^{2^q - p} = B_q^p$. By the hypothesis on p , $A' \sim A_q^p = A$. By Step 4, $B \sim A_q^1 \sim B'$. Thus by the [Domination Lemma](#), $A \cup B \sim A' \cup B'$.

By Step 2, $A \cup B \sim A_q^{p+1}$, and by definition, $A' \cup B' = S \setminus A_q^{2^q - (p+1)} = B_q^{p+1}$, so altogether $A_q^{p+1} \sim B_q^{p+1}$. By induction on p , we are done. \square

◦ STEP 6: Define the binary operation \uplus .

For each $q \in \{0, 1, \dots\}$ and each pair $p, p' \in \{0, 1, \dots, 2^q\}$ such that $p + p' \leq 2^q$, define $A_q^p \uplus A_q^{p'} \in \mathcal{A}$ by:

$$A_q^p \uplus A_q^{p'} \equiv \mathcal{G}^S(A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'})). \quad \square$$

◦ STEP 7: For each $q \in \{0, 1, \dots\}$ and each pair $p, p' \in \{0, 1, \dots, 2^q\}$ such that $p + p' \leq 2^q$,

$$A_q^p \uplus A_q^{p'} = A_q^{p+p'}.$$

Let $q \in \{0, 1, \dots\}$ and let $p \in \{0, 1, \dots, 2^q\}$. We proceed by induction on p' . For the base step, let $p' = 0$. Then by the [Idempotence Lemma](#), $A_q^p \uplus A_q^{p'} = \mathcal{G}^S(A_q^p) = A_q^p$.

For the inductive hypothesis, assume $p' \in \{0, 1, \dots, [2^q - 1] - p\}$ is such that $A_q^p \uplus A_q^{p'} = A_q^{p+p'}$. Define $A, B, C \in \mathcal{A}$ by:

$$\begin{aligned} A &\equiv A_q^p, \\ B &\equiv \mathcal{G}^{S \setminus A}(A_q^{p'}), \text{ and} \\ C &\equiv \mathcal{G}^{S \setminus (A \cup B)}(A_q^1). \end{aligned}$$

By hypothesis, $A_q^{p+p'} = A_q^p \uplus A_q^{p'} = \mathcal{G}^S(A \cup B)$. Since \succsim satisfies 1-AS and, by *monotonicity*, $S \succsim A \cup B$, thus by the [1-AS Lemma](#), $\mathcal{G}^S(A \cup B) \sim A \cup B$. Altogether, $A_q^{p+p'} \sim A \cup B$.

By *monotonicity* and Step 4, $S \setminus A_q^{p+p'} \succsim \mathcal{G}^{S \setminus A_q^{p+p'}}(A_q^1) \sim A_q^1$, so by the [Complement Lemma](#), $S \setminus (A \cup B) \sim S \setminus A_q^{p+p'} \succsim A_q^1$. By Step 3, $\succsim|_{S \setminus A}$ satisfies 2-AS, so by the [Greedy Removal Lemma](#), $\succsim_{(S \setminus A) \setminus B} = \succsim_{S \setminus (A \cup B)}$ satisfies 1-AS. Thus by the [1-AS Lemma](#), $A_q^1 \sim C$. By Step 4, $\mathcal{G}^{S \setminus A_q^{p+p'}}(A_q^1) \sim A_q^1$, so $\mathcal{G}^{S \setminus A_q^{p+p'}}(A_q^1) \sim C$.

Since $A_q^{p+p'} \sim A \cup B$ and $\mathcal{G}^{S \setminus A_q^{p+p'}}(A_q^1) \sim C$, thus by Step 2 and the [Domination Lemma](#),

$$\begin{aligned} A_q^{p+p'+1} &\sim A_q^{p+p'} \cup \mathcal{G}^{S \setminus A_q^{p+p'}}(A_q^1) \\ &\sim (A \cup B) \cup C. \end{aligned}$$

By Step 5, $S \setminus A_q^p \sim A_q^{2^q-p}$. Since $2^q - p \geq p' + 1$, by Step 2 and *monotonicity*, $A_q^{2^q-p} \succsim A_q^{p'+1}$, so $S \setminus A_q^p \succsim A_q^{p'+1}$. By Step 3, $\succsim|_{S \setminus A_q^p}$ satisfies 2-AS, so by the [1-AS Lemma](#), $\mathcal{G}^{S \setminus A_q^p}(A_q^{p'+1}) \sim A_q^{p'+1}$.

Since $S \setminus A_q^p \succsim A_q^{p'+1}$, by Step 2 and *monotonicity*, $S \setminus A_q^p \succsim A_q^{p'}$. By Step 3, $\succsim|_{S \setminus A_q^p}$ satisfies 2-AS, so by the [1-AS Lemma](#), $A_q^{p'} \sim B$. By Step 4, $\mathcal{G}^{S \setminus A_q^{p'}}(A_q^1) \sim A_q^1$, and as argued above, $A_q^1 \sim C$, so $\mathcal{G}^{S \setminus A_q^{p'}}(A_q^1) \sim C$. Thus by Step 2 and the [Domination Lemma](#), $A_q^{p'+1} \sim A_q^{p'} \cup \mathcal{G}^{S \setminus A_q^{p'}}(A_q^1) \sim B \cup C$.

Altogether, $\mathcal{G}^{S \setminus A_q^p}(A_q^{p'+1}) \sim B \cup C$. Since \succsim satisfies 1-AS and, by *monotonicity*, $S \succsim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'+1})$, thus by the [1-AS Lemma](#) and *separability*,

$$\begin{aligned} A_q^p \biguplus A_q^{p'+1} &= \mathcal{G}^S(A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'+1})) \\ &\sim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'+1}) \\ &\sim A_q^p \cup (B \cup C) \\ &= A \cup (B \cup C). \end{aligned}$$

Since $(A \cup B) \cup C = A \cup (B \cup C)$, thus $A_q^{p+p'+1} \sim A_q^p \biguplus A_q^{p'+1}$. By the [Idempotence Lemma](#), $A_q^{p+p'+1} = A_q^p \biguplus A_q^{p'+1}$.

By induction on p' , for each $p' \in \{0, 1, \dots, 2^q - 1\}$, $A_q^p \biguplus A_q^{p'} = A_q^{p+p'}$. Since $q \in \{0, 1, \dots\}$ and $p \in \{0, 1, \dots, 2^q\}$ were arbitrary, we are done. \square

◦ STEP 8: $\lim A_q^1 = \emptyset$.

By Step 2, *monotonicity*, and the [Convergence Lemma](#), $(A_q^1) \in \mathcal{A}^{\mathbb{N}}$ is convergent. Assume, by way of contradiction, there is $s \in \lim A_q^1$.

We claim for each $i \in \mathbb{N}$ and each $k \in \mathbb{N}$, $A_i^1 \succ \{s, s+1, \dots, s+k\}$. We proceed by induction on k . For the base step, let $k = 0$. Then by *continuity* and *monotonicity*, for each $i \in \mathbb{N}$, $A_i^1 \succsim \lim A_q^1 \succsim \{s\}$, so by Step 1, $A_i^1 \succ A_{i+1}^1 \succsim \{s\}$.

For the inductive hypothesis, assume $k \in \mathbb{N}$ is such that for each $i \in \mathbb{N}$, $A_i^1 \succ \{s, s+1, \dots, s+k\}$. Let $i \in \mathbb{N}$. By Step 1, $H(A_i^1) \sim A_{i+1}^1$. By Step 4 and Step 1, $H'(A_{i+1}^1) \sim A_{i+1}^1 \sim \mathcal{G}^{S \setminus A_{i+1}^1}(A_{i+1}^1)$. By the inductive hypothesis, $A_{i+1}^1 \succ \{s, s+1, \dots, s+k\}$. By Step 4 and the inductive hypothesis, $\mathcal{G}^{S \setminus A_{i+1}^1}(A_{i+1}^1) \sim A_{i+1}^1 \succ \{s\} \succsim \{s+k+1\}$. Thus by Step 1 and two applications of the [Domination Lemma](#),

$$\begin{aligned} A_i^1 &\sim H(A_i^1) \cup H'(A_i^1) \\ &\sim A_{i+1}^1 \cup \mathcal{G}^{S \setminus A_{i+1}^1}(A_{i+1}^1) \\ &\succ \{s, s+1, \dots, s+k\} \cup \{s+k+1\}. \end{aligned}$$

Since $i \in \mathbb{N}$ was arbitrary, for each $i \in \mathbb{N}$, $A_i^1 \succ \{s, s+1, \dots, s+k+1\}$. By induction on k , we are done.

Thus by *continuity*, for each $i \in \mathbb{N}$, $A_i^1 \succsim \lim\{s, s+1, \dots, s+k\} = \{s, s+1, \dots\}$. If $s = 1$, then by Step 1, $S = A_0^1 \succ A_1^1 \succsim \{1, 2, \dots\} = S$, contradicting $S \sim S$. Thus $s > 1$, so by 2-AS, for each $i \in \mathbb{N}$, $A_i^1 \succ \{s, s+1, \dots\} \succsim \{s-1\}$.

Thus for each $s \in S$, if for each $i \in \mathbb{N}$, $A_i^1 \succ \{s\}$, then (1) $s \neq 1$, and (2) for each $i \in \mathbb{N}$, $A_i^1 \succ \{s-1\}$. Then there can be no such s , contradicting that there is. \square

◦ STEP 9: Define $\{A_v\}_{v \in 2} \subseteq \mathcal{A}$ and $\{\mathcal{Z}_v\}_{v \in 2} \subseteq [\mathcal{A}/\sim]$ such that for each pair $v, v' \in 2$ such that $v' > v$, $A_{v'} \succ A_v$.

Let $v \in 2$. Then there are $p, q \in \{0, 1, \dots\}$ such that $p \leq 2^q$ and $v = \frac{p}{2^q}$. Define

$$A_v \equiv A_q^p, \text{ and } \mathcal{Z}_v \equiv \{A \in \mathcal{A} \mid A \sim A_v\}.$$

By Step 4, this is well-defined.

Let $v, v' \in 2$ such that $v' > v$. Since $v' - v > 0$, by Step 1, Step 2, and *monotonicity*, $A_{v'-v} \succ \emptyset$. Similarly, $A_{1-v} \succ \emptyset$. By Step 5, $S \setminus A_v \sim A_{1-v} \succ \emptyset$, so $\mathcal{G}^{S \setminus A_v}(A_{v'-v}) \succ \emptyset$. Since \succsim satisfies 1-AS and, by *monotonicity*, $S \succsim A_v \cup \mathcal{G}^{S \setminus A_v}(A_{v'-v})$, thus by the [1-AS Lemma](#), $\mathcal{G}^S(A_v \cup \mathcal{G}^{S \setminus A_v}(A_{v'-v})) \sim A_v \cup \mathcal{G}^{S \setminus A_v}(A_{v'-v})$. Thus by Step 6, $A_{v'} = \mathcal{G}^S(A_v \cup \mathcal{G}^{S \setminus A_v}(A_{v'-v})) \sim A_v \cup \mathcal{G}^{S \setminus A_v}(A_{v'-v})$, so by *separability*, $A_{v'} \succ A_v$. \square

◦ STEP 10: For each convergent pair $(v_i), (v'_i) \in 2^{\mathbb{N}}$ such that $\lim v_i = \lim v'_i$, if $(A_{v_i}), (A_{v'_i}) \in \mathcal{A}^{\mathbb{N}}$ are convergent, then $\lim A_{v_i} \sim \lim A_{v'_i}$.

Define $v_\infty \equiv \lim v_i = \lim v'_i$, $A_\infty \equiv \lim A_{v_i}$, and $A'_\infty \equiv \lim A_{v'_i}$. Let $v \in 2$. If $v > v_\infty$, then there is $v' \in 2$ such that $v > v' > v_\infty$. Since $\lim v_i = v_\infty$, there is $i^* \in \mathbb{N}$ such that for each $i \geq i^*$, $v' > v_i$. Thus for each $i \geq i^*$, $A_{v'} \succ A_{v_i}$, so by Step 9 and *continuity*, $A_v \succ A_{v'} \succsim A_\infty$. By the same argument, $v > v_\infty$ implies $A_v \succ A'_\infty$.

Similarly, for each $v \in 2$, $v_\infty > v$ implies $A_\infty \succ A_v$ and $A'_\infty \succ A_v$.

Assume, by way of contradiction, $A_\infty \not\sim A'_\infty$. Assume, without loss of generality, $A_\infty \succ A'_\infty$. Define $G \equiv \mathcal{G}^S(A'_\infty)$. Since \succsim satisfies 1-AS and, by *monotonicity*, $S \succsim A'_\infty$, thus by the [1-AS Lemma](#), $A_\infty \succ A'_\infty \sim G$.

Necessarily $S \setminus G \succ \emptyset$, else by the [Complement Lemma](#) and *monotonicity*, $G \succsim S \succsim A_\infty$, contradicting $A_\infty \succ G$. Then there is $q \in \mathbb{N}$ such that $S \setminus G \succ A_q^1$, else by Step 8 and *continuity*, $\emptyset = \lim A_q^1 \succsim S \setminus G$, contradicting $S \setminus G \succ \emptyset$.

Since \succsim satisfies 3-AS, thus by the [Greedy Removal Lemma](#), $\succsim|_{S \setminus G}$ satisfies 2-AS, so by Step 1 and the [1-AS Lemma](#), for each $q' \in \mathbb{N}$ such that $q' \geq q$, $\mathcal{G}^{S \setminus G}(A_{q'}^1) \sim A_{q'}^1$. Then by Step 1, $(\mathcal{G}^{S \setminus G}(A_{q'}^1)) \in \mathcal{A}^{\mathbb{N}}$ is monotonic, so by the [Convergence Lemma](#), it is convergent. By the [Limit-Order Lemma](#) and Step 8, $\lim \mathcal{G}^{S \setminus G}(A_{q'}^1) \sim \lim A_{q'}^1 = \emptyset$, so $\lim \mathcal{G}^{S \setminus G}(A_{q'}^1) = \emptyset$. Then $\lim G \cup \mathcal{G}^{S \setminus G}(A_{q'}^1) = G$. Thus there is $q^* \in \mathbb{N}$ such that (1) $q^* \geq q$ and thus $\mathcal{G}^{S \setminus G}(A_{q^*}^1) \sim A_{q^*}^1$, and (2) $A_\infty \succ G \cup \mathcal{G}^{S \setminus G}(A_{q^*}^1)$, else by *continuity* $G \succsim A_\infty$, contradicting $A_\infty \succ G$.

We proceed by reaching a contradiction in three cases:

CASE 1: $v_\infty = 0$. Then since $\frac{1}{2^{q^*}} > v_\infty$, by *monotonicity*, $A_{q^*}^1 \succ A_\infty \succ G \cup \mathcal{G}^{S \setminus G}(A_{q^*}^1) \succsim \mathcal{G}^{S \setminus G}(A_{q^*}^1) \sim A_{q^*}^1$, contradicting $A_{q^*}^1 \sim A_{q^*}^1$.

CASE 2: $v_\infty = 1$. Since $v_\infty > \frac{2^{q^*} - 1}{2^{q^*}}$, $G \sim A'_\infty \succ A_{q^*}^{2^{q^*}-1}$. Since $\mathcal{G}^{S \setminus G}(A_{q^*}^1) \sim A_{q^*}^1 \lesssim \mathcal{G}^{S \setminus A_{q^*}^{2^{q^*}-1}}(A_{q^*}^1)$, thus by the [Domination Lemma](#), $A_\infty \succ G \cup \mathcal{G}^{S \setminus G}(A_{q^*}^1) \succ A_{q^*}^{2^{q^*}-1} \cup \mathcal{G}^{S \setminus A_{q^*}^{2^{q^*}-1}}(A_{q^*}^1)$. Since \lesssim satisfies 1-AS and, by *monotonicity*, $S \lesssim A_{q^*}^{2^{q^*}-1} \cup \mathcal{G}^{S \setminus A_{q^*}^{2^{q^*}-1}}(A_{q^*}^1)$, thus by the [1-AS Lemma](#), $A_{q^*}^{2^{q^*}-1} \cup \mathcal{G}^{S \setminus A_{q^*}^{2^{q^*}-1}}(A_{q^*}^1) \sim A_{q^*}^{2^{q^*}-1} \uplus A_{q^*}^1$. But then by Step 7, Step 4, and Step 1, $A_\infty \succ A_{q^*}^{2^{q^*}-1} \uplus A_{q^*}^1 = A_{q^*}^{2^{q^*}} = S$, contradicting $S \lesssim A_\infty$.

CASE 3: $v_\infty \in (0, 1)$. Define $\epsilon^* \equiv \frac{1}{2^{q^*}}$. Since $\mathbb{2}$ is dense in $[0, 1]$, there is $v^* \in \mathbb{2}$ such that $1 \geq v^* + \epsilon^* > v_\infty > v^*$. Then $A_{v^*+\epsilon^*} \succ A_\infty \succ G \sim A'_\infty \succ A_{v^*}$.

Since $1 - v^* \geq \epsilon^*$, thus by Step 5, $S \setminus A_{v^*} \sim A_{1-v^*} \lesssim A_{\epsilon^*} = A_{q^*}^1$. Then by Step 3 and the [1-AS Lemma](#), $\mathcal{G}^{S \setminus G}(A_{q^*}^1) \sim A_{q^*}^1 \sim \mathcal{G}^{S \setminus A_{v^*}}(A_{q^*}^1)$. Since $G \succ A_{v^*}$, thus by the [Domination Lemma](#), $A_\infty \succ G \cup \mathcal{G}^{S \setminus G}(A_{q^*}^1) \succ A_{v^*} \cup \mathcal{G}^{S \setminus A_{v^*}}(A_{q^*}^1)$.

Since \lesssim satisfies 1-AS and, by *monotonicity*, $S \lesssim A_{v^*} \cup \mathcal{G}^{S \setminus A_{v^*}}(A_{q^*}^1)$, thus by the [1-AS Lemma](#), $A_{v^*} \uplus A_{\epsilon^*} = A_{v^*} \cup \mathcal{G}^{S \setminus A_{v^*}}(A_{q^*}^1)$. But then by Step 7, $A_\infty \succ A_{v^*} \uplus A_{\epsilon^*} = A_{v^*+\epsilon^*}$, contradicting $A_{v^*+\epsilon^*} \succ A_\infty$. \square

◦ STEP 11: Conclude.

We verify that $\{\mathcal{Z}_v\}_{v \in \mathbb{2}}$ satisfies [SC1], [SC2], [SC3], and [SC4].

SC1: By Step 1 and Step 9, $\emptyset = A_0 \in \mathcal{Z}_0$ and $S = A_1 \in \mathcal{Z}_1$.

SC2: Let $v, v' \in \mathbb{2}$ such that $v + v' \leq 1$. Then there are $p, p', q \in \{0, 1, \dots\}$ such that $v = \frac{p}{2^q}$ and $v' = \frac{p'}{2^q}$, and $p + p' \leq 2^q$.

By construction, $A_q^p \in \mathcal{Z}_v$, $A_q^{p'} \in \mathcal{Z}_{v'}$, and $A_q^{p+p'} \in \mathcal{Z}_{v+v'}$. By Step 5, $S \setminus A_q^p \sim A_q^{2^q-p}$. Since $p + p' \leq 2^q$, thus by *monotonicity* and Step 2, $S \setminus A_q^p \lesssim A_q^{p'}$. By Step 3, $\lesssim|_{S \setminus A_q^p}$ satisfies 2-AS, so by the [1-AS Lemma](#),

$$\mathcal{G}^{S \setminus A_q^p}(A_q^{p'}) \sim A_q^{p'}.$$

Thus we have disjoint $A_q^p \in \mathcal{Z}_v$ and $\mathcal{G}^{S \setminus A_q^p}(A_q^{p'}) \in \mathcal{Z}_{v'}$.

Since \lesssim satisfies 1-AS and, by *monotonicity*, $S \lesssim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'})$, thus by the [1-AS Lemma](#), $\mathcal{G}^S(A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'})) \sim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'})$. Thus by Step 7,

$$\begin{aligned} A_q^{p+p'} &= A_q^p \uplus A_q^{p'} \\ &= \mathcal{G}^S(A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'})) \\ &\sim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'}), \end{aligned}$$

so $A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'}) \in \mathcal{Z}_{v+v'}$.

SC3: Let $(v_i^+) \in 2^\mathbb{N}$ be non-increasing and let $(v_i^+) \in 2^\mathbb{N}$ be non-decreasing such that $\lim v_i^+ = \lim v_i^-$.

- (i) for each $i \in \mathbb{N}$, $v_i + w_i \leq 1$ and $v_i' + w_i' \leq 1$, and
- (ii) $\lim v_i = \lim v_i'$ and $\lim w_i = \lim w_i'$.

For each $i \in \mathbb{N}$, define $A_i \equiv A_{v_i}$ and $A'_i \equiv A_{v'_i}$. Then for each $i \in \mathbb{N}$, $A_i \in \mathcal{Z}_{v_i}$, $A'_i \in \mathcal{Z}_{v'_i}$. By Step 9, (A_i) and (A'_i) are monotonic, so by the [Convergence Lemma](#), (A_i) and (A'_i) are convergent, so by Step 10, $\lim A_i \sim \lim A'_i$.

SC4: Let $(v_i), (w_i) \in \mathcal{P}^{\mathbb{N}}$ be monotonic such that for each $i \in \mathbb{N}$, $v_i + w_i \leq 1$. For each $i \in \mathbb{N}$, define $A_i \equiv A_{v_i}$ and $B_i \equiv \mathcal{G}^{S \setminus A_i}(A_{w_i})$.

By Step 5, for each $i \in \mathbb{N}$, $S \setminus A_i \sim A_{1-v_i}$, so since $1 - v_i \geq w_i$, by Step 9, $S \setminus A_i \succsim A_{w_i}$. Thus by Step 3 and the [1-AS Lemma](#), for each $i \in \mathbb{N}$, $B_i \sim A_{w_i}$. Altogether, for each $i \in \mathbb{N}$, A_i and B_i are disjoint with $A_i \in \mathcal{Z}_{v_i}$ and $B_i \in \mathcal{Z}_{w_i}$.

By Step 9, (A_i) and (B_i) are monotonic, and by construction, for each $i \in \mathbb{N}$, A_i and B_i are disjoint; thus by the [Convergence Lemma](#), $(\mathcal{G}_1^S(A_i, B_i))$ and $(\mathcal{G}_2^S(A_i, B_i))$ are convergent. By the [Idempotence Lemma](#), these are, respectively (A_i) and (B_i) . ■

• **THEOREM 1:** If $(S, \mathcal{A}, \succsim)$ is a purely- σ -catalogued qualitative probability space satisfying *(monotone) continuity* and 3-AS, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

PROOF: The result follows immediately from the [Supercabinet Construction Lemma](#) and the [Supercabinet Blueprint Lemma](#). ■

Appendix 6

In this appendix, we prove the [Carving Lemma](#) and [Theorem 2](#).

[Theorem KK](#) is an implication of the cardinal analogue of the [1-AS Lemma](#) ([Takeya, 1914](#); [Takeya, 1915](#)) and the cardinal analogue of the [Greedy Removal Lemma](#) ([Komornik, 2015](#)). It provides conditions for a sequence of numbers to have k disjoint subsequences whose sums correspond to a given list v_1, v_2, \dots, v_k :

DEFINITION: A sequence $(\mu_i) \in [0, 1]^{\mathbb{N}}$ is k -*Takeya* if

- (i) (μ_i) is non-increasing,
- (ii) $\lim \mu_i = 0$, and
- (iii) for each $i \in \mathbb{N}$, $\sum_{j>i} \mu_j \geq k\mu_i$.

• **THEOREM KK** ([TAKEYA, 1914](#); [TAKEYA, 1915](#); [KOMORNIK, 2015](#)): For each k -Takeya $(\mu_i) \in [0, 1]^{\mathbb{N}}$ and each $v_1, v_2, \dots, v_k \in [0, \sum \mu_i]$ such that $v_1 + v_2 + \dots + v_k \leq \sum \mu_i$, there are pairwise disjoint $A_1, A_2, \dots, A_k \subseteq \mathbb{N}$ such that for each $i \in \{1, 2, \dots, k\}$,

$$\sum_{A_k} \mu_i = v_k.$$

The [Carving Lemma](#) states that if a first event that contains no atoms is at least as likely as a second event, then the first contains a subevent that is as likely as the second.

• **CARVING LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a catalogued qualitative probability space satisfying *(monotone) continuity*, then for each pair $A, B \in \mathcal{A}$ such that $A \in \mathcal{A}^\circ$ and $A \succsim B$, there is $B' \subseteq A$ such that $B' \in \mathcal{A}$ and $B' \sim B$.

PROOF: Let $(S, \mathcal{A}, \succsim)$, A, B satisfy the hypotheses. If $A \sim B$, then define $B' \equiv A$ and we are done; if $\emptyset \sim B$, then define $B' \equiv \emptyset$ and we are done; thus assume $A \succ B \succ \emptyset$. By *monotonicity*, $S^\circ \succsim A \succ \emptyset$.

For each pair $C, C' \in \mathcal{A}^\circ$, define $C \succsim^\circ C'$ if and only if $C \succsim C'$. Since $S^\circ \succ \emptyset$, it is straightforward to verify that $(S^\circ, \mathcal{A}^\circ, \succsim^\circ)$ is an atomless qualitative probability space satisfying *monotone continuity*, so by [Theorem V2](#), it has a unique representation $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$ that is infinitely-divisible.

Define $V^+, V^- \subseteq [0, \mu^\circ(A)]$ by:

$$V^+ \equiv \{v \in [0, \mu^\circ(A)] \mid C \in \mathcal{A}^\circ \text{ and } \mu^\circ(C) = v \text{ implies } C \succsim B\}, \text{ and} \\ V^- \equiv \{v \in [0, \mu^\circ(A)] \mid C \in \mathcal{A}^\circ \text{ and } \mu^\circ(C) = v \text{ implies } B \succsim C\}.$$

Since $A \succ B \succ \emptyset$, thus $\mu^\circ(A) \in V^+$ and $0 \in V^-$, so $\inf V^+$ and $\sup V^-$ are well-defined. Since μ° is infinitely-divisible, $V^+ \cup V^- = [0, \mu^\circ(A)]$. Moreover, since μ° represents \succsim° , for each pair $v, v' \in [0, 1]$ (i) $v \in V^+$ and $v' > v$ implies $v' \in V^+$, and (ii) $v \in V^-$ and $v' < v$ implies $v' \in V^-$. Altogether, $\inf V^+ = \sup V^-$; define $v^* \equiv \inf V^+ = \sup V^-$. Since μ° is infinitely-divisible, there is $B' \in \mathcal{A}^\circ$ such that $\mu^\circ(B') = v^*$.

First, assume $v^* \neq \mu^\circ(A)$. Since μ° is infinitely-divisible, there is $(A_i) \in (\mathcal{A}^\circ)^\mathbb{N}$ such that

- (1) for each $i \in \mathbb{N}$, $\mu^\circ(A_i) = v^* + (\frac{1}{2})^i(\mu^\circ(A) - v^*)$, and
- (2) $A_1 \supseteq A_2 \supseteq \dots$

By (2), there is $A_\infty \equiv \lim(A_i)$. For each $i \in \mathbb{N}$, $v_i > v^*$, so $v_i \in V^+$, so $A_i \succsim B$; thus by *continuity*, $A_\infty \succsim B$. Since μ° is σ -additive, by (1) $\mu^\circ(A_\infty) = v^*$, so $B' \sim A_\infty$. Altogether, $v^* \neq 0$ implies $B' \succsim B$.

By a symmetric argument, $v^* \neq 0$ implies $B \succsim B'$. We cannot have that $v^* = 0$, else $\emptyset \sim B' \succsim B$, contradicting $B \succ \emptyset$. Similarly, we cannot have that $v^* = \mu^\circ(A)$, else $B \succsim B' \sim A$, contradicting $A \succ B$. Thus $v^* \in (0, \mu^\circ(A))$, so $B' \sim B$ as desired. ■

• **THEOREM 2:** If $(S, \mathcal{A}, \succsim)$ is a finitely-catalogued qualitative probability space satisfying (*monotone*) *continuity* and 3-*AS*, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

PROOF: If $S^\oplus = \emptyset$, then we are done by [Theorem V2](#), so assume $S^\oplus \neq \emptyset$.

◦ STEP 1: Define μ° .

For each pair $A, B \subseteq S^\circ$, define $A \succsim^\circ B$ if and only if $A \succsim B$. Since \succsim satisfies 3-*AS* and *monotonicity*, thus $S^\circ \succ \emptyset$, so $S^\circ \succ^\circ \emptyset$. It is immediate that $(S^\circ, \mathcal{A}^\circ, \succsim^\circ)$ is an atomless qualitative probability space satisfying *monotone continuity*, so by [Theorem V2](#), it has a unique representation $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$, and μ° is infinitely-divisible.

◦ STEP 2: Construct $(A'_i) \in \mathcal{A}^\mathbb{N}$ partitioning S , and the σ -algebra $\mathcal{A}' \subseteq \mathcal{A}$.

To define $(A'_i) \in \mathcal{A}^\mathbb{N}$, we first define $(A_i) \in \mathcal{A}^\mathbb{N}$. For each $i \in \{1, 2, \dots, |I|\}$, define $A_i \equiv \{i\}$. Since $S \setminus (\cup_{j \leq |I|} A_j) = S^\circ$, thus by 3-*AS* and *separability*, $S \setminus (\cup_{j \leq |I|} A_j) \succ A_{|I|} \succ \emptyset$. Clearly, $S \setminus (\cup_{j \leq |I|} A_j) \in \mathcal{A}^\circ$.

Let $A_{|I|+1} \subseteq S^\circ$ such that $S^\circ \succ A_{|I|} \succ A_{|I|+1} \succ \emptyset$; we can do this by the [Carving Lemma](#) and infinite-divisibility of μ° . By the [Complement Lemma](#), $S \setminus (\cup_{j \leq |I|+1} A_j) = S^\circ \setminus A_{|I|+1} \succ \emptyset$, and clearly, $S \setminus (\cup_{j \leq |I|+1} A_j) \in \mathcal{A}^\circ$.

For the recursive step, assume we have defined $A_1, A_2, \dots, A_i \in \mathcal{A}$ such that

- (i) $(S \setminus \cup_{j \leq i} A_j) \succ \emptyset$, and
- (ii) $(S \setminus \cup_{j \leq i} A_j) \in \mathcal{A}^\circ$.

Since $(S \setminus \cup_{j \leq i} A_j) \subseteq S^\circ$ and μ° is infinitely-divisible, thus $(S \setminus \cup_{j \leq i} A_j)$ can be partitioned into B_1, B_2, B_3, B_4 that are equally-likely, according to both \succsim° and \succsim .

If $A_i \succ B_1$, then define $A_{i+1} \equiv B_1$. Then $(S \setminus \cup_{j \leq i+1} A_j) = B_2 \cup B_3 \cup B_4$, which by *separability* and *monotonicity* is not null. Clearly, $(S \setminus \cup_{j \leq i} A_j) \in \mathcal{A}^\circ$.

If $B_1 \succ A_i$, then by the [Carving Lemma](#), there is $C \subseteq B_1$ such that $C \sim^\circ A_i$, and thus $C \sim A_i$. Then $(S \setminus \cup_{j \leq i+1} A_j)$ contains $B_2 \cup B_3 \cup B_4$, and thus by *separability* and *monotonicity* is not null.

Consider $(A_j)_{j > |I|}$, which by construction are pairwise-disjoint. We cannot have that $A_{|I|+1} \sim A_{|I|+2} \sim \dots$, else $\mu^\circ(A_{|I|+1}) = \mu^\circ(A_{|I|+2}) = \dots$, so $\mu^\circ(\cup_{j > |I|} A_j) > 1$, contradicting that $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$. Thus there is some smallest $i^* \in \{|I| + 1, |I| + 2, \dots\}$ such that $A_{i^*} \succ A_{i^*+1}$. By construction, $\mu^\circ(A_{i^*+1}) = \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i^*} A_j)$. Moreover, for each $i \geq i^*$ such that $\mu^\circ(A_{i+1}) = \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i} A_j)$, by construction $A_{i+1} \succ A_{i+2}$ and $\mu^\circ(A_{i+2}) = \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i+1} A_j)$. Altogether, there is $i^* \in \{|I| + 1, |I| + 2, \dots\}$ such that for each $i \geq i^*$, $\mu^\circ(A_{i+1}) = \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i} A_j)$. It follows from the additivity of μ° that

$$\begin{aligned}
\sum_{j > i^*} \mu^\circ(A_j) &= \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i^*} A_j) + \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i^*+1} A_j) + \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i^*+2} A_j) + \dots \\
&= \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i^*} A_j) + \frac{3}{4}\frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i^*} A_j) + \left(\frac{3}{4}\right)^2 \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i^*} A_j) + \dots \\
&= \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j \leq i^*} A_j) \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k \\
&= \mu^\circ(S^\circ \setminus \cup_{j \leq i^*} A_j)
\end{aligned}$$

Since $j \leq |I|$ implies $A_j \cap S^\circ = \emptyset$, thus

$$\sum_{j > i^*} \mu^\circ(A_j) = \mu^\circ(S^\circ \setminus \cup_{i^* \geq j > |I|} A_j).$$

Moreover, since (A_i) is pairwise-disjoint and since $j > |I|$ implies $A_j \subseteq S^\circ$, thus by additivity of μ° ,

$$\begin{aligned}
\mu^\circ(S^\circ \setminus \cup A_j) &= \mu^\circ(S^\circ) - \sum_{j > |I|} \mu^\circ \\
&= \mu^\circ(S^\circ) - \left[\sum_{i^* \geq j > |I|} \mu^\circ(A_j) + \sum_{j > i^*} \mu^\circ(A_j) \right] \\
&= \mu^\circ(S^\circ) - [\mu^\circ(\cup_{i^* \geq j > |I|} A_j) + \mu^\circ(S^\circ \setminus \cup_{i^* \geq j > |I|} A_j)] \\
&= \mu^\circ(S^\circ) - \mu^\circ(S^\circ) = 1 - 1 = 0.
\end{aligned}$$

Since μ° represents \succsim° , thus $(S^\circ \setminus \cup A_j)$ is null, according to both \succsim° and \succsim .

Define $A'_1 \equiv A_1 \cup (S^\circ \setminus \cup A_j)$, and for each $i \in \mathbb{N} \setminus \{1\}$, define $A'_i \equiv A_i$. By *separability*, for each $i \in \mathbb{N}$, $A'_i \sim A_i$. By construction, (A'_i) partitions S , with $(A'_i)_{i \leq |I|}$ partitioning S^\oplus and $(A'_i)_{i > |I|}$ partitioning S° .

Define $\mathcal{A}' \subseteq \mathcal{A}$ to be the σ -algebra generated by (A'_i) : $A' \in \mathcal{A}'$ if and only if there is $J \subseteq \mathbb{N}$ such that $A' = \cup_{i \in J} A'_i$.

◦ STEP 3: $(\mu^\circ(A'_i))_{i > |I|}$ is 3-Kekeya.

Throughout this step, we use the index set $\{|I|+1, |I|+2, \dots\}$, and we suppress it for convenience. By construction, (A'_i) is non-increasing in likelihood, so since μ° represents \succsim° , thus $(\mu^\circ(A'_i))$ is non-increasing. Moreover, $(\mu^\circ(A'_i))$ is bounded, so it is convergent. We cannot have $\lim \mu^\circ(A'_i) > 0$, else $\sum \mu^\circ(A'_i) = \infty$, which since μ° is σ -additive implies $\mu^\circ(S^\circ) = \mu^\circ(\cup A'_i) = \sum \mu^\circ(A'_i) = \infty$, contradicting $\mu^\circ(S^\circ) = 1$. Thus $\lim \mu^\circ(A'_i) = 0$.

Let $i > |I|$. By construction, $\mu^\circ(A'_i) \leq \frac{1}{4}\mu^\circ(S^\circ \setminus \cup_{j < i} A'_j)$. Since (A'_i) partitions S° , $\mu^\circ(S^\circ \setminus \cup_{j < i} A'_j) = \mu^\circ(\cup_{j \geq i} A'_j)$, so by σ -additivity of μ° , $\mu^\circ(A'_i) \leq \frac{1}{4}(\mu^\circ(A'_i) + \sum_{j > i} \mu^\circ(A'_j))$. Rearranging, $\sum_{j > i} \mu^\circ(A'_j) \geq 3\mu^\circ(A'_i)$. Since $i > |I|$ was arbitrary, we are done. \square

◦ STEP 4: Define $(S^*, \mathcal{A}^*, \succsim^*)$, $\Psi^* : \mathcal{A}^* \rightarrow \mathcal{A}'$, and $\Psi' : \mathcal{A}' \rightarrow \mathcal{A}^*$.

Define $S^* \equiv \{1, 2, \dots\}$ and define $\mathcal{A}^* \equiv 2^{S^*}$. Define $\Psi^* : \mathcal{A}^* \rightarrow \mathcal{A}'$ as follows: for each $A^* \in \mathcal{A}^*$,

$$\Psi^*(A^*) = \cup_{i \in A^*} A'_i.$$

For each pair $A, B \in \mathcal{A}^*$, define

$$A \succsim^* B \text{ if and only if } \Psi^*(A) \succsim \Psi^*(B).$$

Clearly, Ψ^* is a bijection that preserves disjoint unions: if $A, B \in \mathcal{A}^*$ are disjoint, then $\Psi^*(A), \Psi^*(B) \in \mathcal{A}'$ are disjoint.

Let $\Psi' : \mathcal{A}' \rightarrow \mathcal{A}^*$ be the inverse of Ψ^* ; clearly, Ψ' preserves disjoint unions.

◦ STEP 5: Verify $(S^*, \mathcal{A}^*, \succsim^*)$ is a purely-catalogued qualitative probability space satisfying *monotone continuity* and *3-AS*.

We verify each property in sequence:

ORDER: To see that \succsim^* is complete, let $A, B \in \mathcal{A}^*$. Since \succsim is complete, $\Psi^*(A) \succsim \Psi^*(B)$; thus $A \succsim^* B$ as desired.

To see that \succsim^* is transitive, let $A, B, C \in \mathcal{A}^*$ such that $A \succsim^* B$ and $B \succsim^* C$. Then $\Psi^*(A) \succsim \Psi^*(B)$ and $\Psi^*(B) \succsim \Psi^*(C)$, so by transitivity of \succsim , $\Psi^*(A) \succsim \Psi^*(C)$; thus $A \succsim^* C$ as desired.

SEPARABILITY: Let $A, B, C \in \mathcal{A}^*$ such that $A \cap C = B \cap C = \emptyset$. Since Ψ^* preserves disjoint pairs, $\Psi^*(A) \cap \Psi^*(C) = \Psi^*(B) \cap \Psi^*(C) = \emptyset$. Since \succsim is *separable*, $\Psi^*(A) \succsim \Psi^*(B)$ if and only if $\Psi^*(A) \cup \Psi^*(C) \succsim \Psi^*(B) \cup \Psi^*(C)$; thus $A \succsim^* B$ if and only if $A \cup C \succsim^* B \cup C$, as desired.

MONOTONICITY: Let $A, B \in \mathcal{A}$ with $A \subseteq B$. Then $\Psi^*(A) \subseteq \Psi^*(B)$. Since \succsim is *monotonic*, $\Psi^*(B) \succsim \Psi^*(A)$; thus $B \succsim^* A$, as desired.

NONDEGENERACY: Since (A_i) partitions S , thus $\Psi^*(S^*) = S$, and clearly $\Psi^*(\emptyset) = \emptyset$. Since \succsim is *monotonic* and *nondegenerate*, $S \succ \emptyset$, so $\Psi^*(S^*) \succ \Psi^*(\emptyset)$, so $S^* \succ^* \emptyset$, as desired.

CONTINUITY: Let $A \in \mathcal{A}^*$ and let $(B_i) \in (\mathcal{A}^*)^\mathbb{N}$ be convergent. Define $B \equiv \lim B_i$. Clearly, $(\Psi^*(B_i))$ is convergent with $\Psi^*(B) = \lim \Psi^*(B_i)$.

If for each $i \in \mathbb{N}$, $B_i \succ^* A$, then for each $i \in \mathbb{N}$, $\Psi^*(B_i) \succ \Psi^*(A)$. Since \succ is continuous, $\Psi^*(B) \succ \Psi^*(A)$; thus $B \succ^* A$, as desired.

If for each $i \in \mathbb{N}$, $A \succ^* B_i$, then by a similar argument $A \succ^* B$, as desired.

3-AS: Let $A \in \mathcal{A}^*$ be an atom. By construction, *separability*, and *monotonicity*, there is $s^* \in S^*$ such that $A = \{s^*\}$. Then $\Psi^*(\{s^*\}) = A'_{s^*}$.

CASE 1: $s^* \leq |I|$. Then A'_{s^*} is an atom. Since \succ satisfies 3-AS, there are

(i) pairwise-disjoint $\mathcal{B} \subseteq (\downarrow A'_{s^*})$, and

(ii) $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \subseteq \mathcal{B}'$ partitioning \mathcal{B} ,

such that for each $i \in \{1, 2, 3\}$, $(\cup \mathcal{B}_i) \succ \Psi^*(A)$.

For each $i \in \{1, 2, 3\}$, define $B_i^\oplus \equiv (\cup \mathcal{B}_i) \cap S^\oplus$, define $B_i^\circ \equiv (\cup \mathcal{B}_i) \cap S^\circ$, and define $v_i \equiv \mu^\circ(B_i)$. Since $\mu^\circ(B_1 \cup B_2 \cup B_3) \leq 1$ and μ° is additive, thus $v_1 + v_2 + v_3 \leq 1 = \mu^\circ(S^\circ) = \sum_{i > |I|} \mu^\circ(A'_i)$. By Step 3 and [Theorem KK](#), there are disjoint $J_1, J_2, J_3 \subseteq \mathbb{N}$ such that for each $i \in \{1, 2, 3\}$, $\sum_{j \in J_i} \mu^\circ(A'_j) = v_i$.

For each $i \in \{1, 2, 3\}$, define

$$\begin{aligned} \mathcal{B}'_i &\equiv \{A'_j \in \mathcal{A}' \mid A'_j \in \mathcal{B}_i^\oplus \text{ or } j \in J_i\}, \\ \mathcal{B}' &\equiv \cup \mathcal{B}'_i, \\ \mathcal{B}^*_i &\equiv \{\Psi'(B) \mid B \in \mathcal{B}'_i\}, \text{ and} \\ \mathcal{B}^* &\equiv \cup \mathcal{B}^*_i. \end{aligned}$$

By construction, \mathcal{B}' is pairwise-disjoint. Let $B \in \mathcal{B}'$. If $B \in \cup \mathcal{B}_i^\oplus$, then $B \in (\downarrow A'_{s^*})$ by construction. If there is $j \in \cup J_i$ such that $B = A'_j$, then $A'_{s^*} \succ A'_{|I|} \succ A'_{|I|+1} \succ A'_j = B$, so $B \in (\downarrow A'_{s^*})$. Altogether, $\mathcal{B}' \subseteq (\downarrow A'_{s^*})$. Clearly, $\mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$ partition \mathcal{B}' .

Let $i \in \{1, 2, 3\}$, define $B'_i \equiv \cup_{j \in J_i} A'_j$. Since μ° is σ -additive, $\mu^\circ(B'_i) = v_i$, so $B'_i \sim B_i$, so by the [Domination Lemma](#) twice, $(\cup \mathcal{B}'_i) = B_i^\oplus \cup B'_i \sim B_i^\oplus \cup B_i^\circ = (\cup \mathcal{B}_i)$; thus $(\cup \mathcal{B}'_i) \succ A'_{s^*}$.

Since Ψ' preserves disjoint pairs, thus (i) \mathcal{B}^* is pairwise-disjoint, (ii) $\mathcal{B}^* \subseteq (\downarrow \{s^*\})$ (for \succ^*), (iii) $\mathcal{B}^*_1, \mathcal{B}^*_2, \mathcal{B}^*_3$ partition \mathcal{B}^* , and (iv) for each $i \in \{1, 2, 3\}$, $(\cup \mathcal{B}^*_i) \succ^* A$. Since A was an arbitrary atom, we are done.

CASE 2: $s^* > |I|$. By Step 3, there is $s^{**} \geq s^*$ such that $\mu^\circ(A_{s^*}) = \mu^\circ(A'_{s^{**}}) > \mu^\circ(A'_{s^{**}+1})$. By Step 3 and [Theorem KK](#), there are disjoint $J_1, J_2, J_3 \subseteq \{s^{**}, s^{**} + 1, \dots\}$ such that for each $i \in \{1, 2, 3\}$, $\sum_{j \in J_i} \mu^\circ(A'_j) = \mu^\circ(A'_{s^*})$.

For each $i \in \{1, 2, 3\}$, define

$$\begin{aligned} \mathcal{B}'_i &\equiv \{A'_j \in \mathcal{A}' \mid j \in J_i\}, \\ \mathcal{B}' &\equiv \cup \mathcal{B}'_i, \\ \mathcal{B}^*_i &\equiv \{\Psi'(B) \mid B \in \mathcal{B}'_i\}, \text{ and} \\ \mathcal{B}^* &\equiv \cup \mathcal{B}^*_i. \end{aligned}$$

It is straightforward to verify that $\mathcal{B}^*, \mathcal{B}^*_1, \mathcal{B}^*_2, \mathcal{B}^*_3$ are as desired, using a simpler argument than that of the previous case. Since A was an arbitrary atom, we are done. \square

PURELY-CATALOGUED: Trivial.

◦ STEP 6: Conclude.

By Step 4 and the [Supercabinet Construction Lemma](#), $(S^*, \mathcal{A}^*, \succsim^*)$ has a supercabinet $\{\mathcal{Z}_v^*\}$. For each $v \in 2$, let $Z_v^* \in \mathcal{Z}_v^*$, and define \mathbb{Z}_v to be the equivalence class of events in \mathcal{A} for $\cup_{i \in Z_v^*} A_i$. It is straightforward to verify $\{\mathbb{Z}_v\}$ is a supercabinet for $(S, \mathcal{A}, \succsim)$, so by the [Supercabinet Blueprint Lemma](#) we are done. ■

Appendix 7

In this appendix, we prove [Theorem 3](#).

• **THEOREM 3:** If $(S, \mathcal{A}, \succsim)$ is a catalogued qualitative probability space satisfying *monotone continuity* and *unlikely cores*, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

PROOF: For each pair $A, B \in \mathcal{A}^\circ$, define $A \succsim^\circ B$ if and only if $A \succsim B$. Similarly, for each pair $A, B \in \mathcal{A}^\oplus$, define $A \succsim^\oplus B$ if and only if $A \succsim B$.

◦ **STEP 1:** Define $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$, $\mu^\oplus : \mathcal{A}^\oplus \rightarrow [0, 1]$, and $\mu : \mathcal{A} \rightarrow [0, 1]$.

Since $S^\circ \succ S^\oplus$, by *monotonicity* $S^\circ \succ \emptyset$. It is immediate that $(S^\circ, \mathcal{A}^\circ, \succsim^\circ)$ is an atomless qualitative probability space satisfying *monotone continuity*; thus by [Theorem V2](#), it has unique representation $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$, and moreover μ° is infinitely-divisible. If $S^\oplus = \emptyset$, we are done, so assume $S^\oplus \neq \emptyset$.

To define $\mu^\oplus : \mathcal{A}^\oplus \rightarrow [0, 1]$, let $A^\oplus \in \mathcal{A}^\oplus$. By *monotonicity*, $S^\circ \succsim S^\oplus \succsim A^\oplus$; thus by the [Carving Lemma](#), there is $A^\circ \in \mathcal{A}^\circ$ such that $A^\oplus \sim A^\circ$. Define $\mu^\oplus(A^\oplus) \equiv \mu^\circ(A^\circ)$.

Define the constant for normalization $\kappa \equiv \mu^\oplus(S^\oplus) + \mu^\circ(S^\circ)$; then $\kappa > 0$. For each $A \in \mathcal{A}$, define

$$\mu(A) \equiv \frac{\mu^\oplus(A \cap S^\oplus) + \mu^\circ(A \cap S^\circ)}{\kappa}. \quad \square$$

◦ **STEP 2:** For each disjoint pair $A^\oplus, B^\oplus \in \mathcal{A}^\oplus$, there are disjoint $A^\circ, B^\circ \in \mathcal{A}^\circ$ such that $A^\oplus \sim A^\circ$ and $B^\oplus \sim B^\circ$.

Let $A^\oplus, B^\oplus \in \mathcal{A}^\oplus$ be disjoint. By *monotonicity*, $S^\circ \succsim S^\oplus \succsim A^\oplus$; thus by the [Carving Lemma](#), there is $A^\circ \in \mathcal{A}^\circ$ such that $A^\oplus \sim A^\circ$. By *separability*, $A^\oplus \cup B^\oplus \sim A^\circ \cup B^\oplus$.

Define $C^\circ \equiv S^\circ \setminus A^\circ$. By *monotonicity* and the above observation,

$$\begin{aligned} A^\circ \cup C^\circ &= S^\circ \\ &\succsim S^\oplus \\ &\succsim A^\oplus \cup B^\oplus \\ &\sim A^\circ \cup B^\oplus, \end{aligned}$$

so by *separability*, $C^\circ \succsim B^\oplus$. Thus by the [Carving Lemma](#), there is $B^\circ \subseteq C^\circ$ such that $B^\circ \in \mathcal{A}^\circ$ and $B^\oplus \sim B^\circ$. By construction, A° and B° are disjoint, as desired. □

◦ **STEP 3:** μ is additive.

Let $A, B \in \mathcal{A}$ be disjoint. We wish to show $\mu(A \cup B) = \mu(A) + \mu(B)$. Define $A^\oplus \equiv A \cap S^\oplus$, $A^\circ \equiv A \cap S^\circ$, $B^\oplus \equiv B \cap S^\oplus$, and $B^\circ \equiv B \cap S^\circ$. By definition,

$$\begin{aligned}\mu(A \cup B) &= \frac{\mu^\oplus(A^\oplus \cup B^\oplus) + \mu^\circ(A^\circ \cup B^\circ)}{\kappa}, \\ \mu(A) &= \frac{\mu^\oplus(A^\oplus) + \mu^\circ(A^\circ)}{\kappa}, \text{ and} \\ \mu(B) &= \frac{\mu^\oplus(B^\oplus) + \mu^\circ(B^\circ)}{\kappa}.\end{aligned}$$

Since $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$, thus $\mu^\circ(A^\circ \cup B^\circ) = \mu^\circ(A^\circ) + \mu^\circ(B^\circ)$, so it suffices to show $\mu^\oplus(A^\oplus \cup B^\oplus) = \mu^\oplus(A^\oplus) + \mu^\oplus(B^\oplus)$.

Since A^\oplus and B^\oplus are disjoint, by Step 2, there are disjoint $A', B' \in \mathcal{A}^\circ$ such that $A^\oplus \sim A'$ and $B^\oplus \sim B'$. By the [Domination Lemma](#) twice, $A^\oplus \cup B^\oplus \sim A' \cup B'$. Thus by construction of μ^\oplus ,

$$\begin{aligned}\mu^\oplus(A^\oplus \cup B^\oplus) &= \mu^\circ(A' \cup B'), \\ \mu^\oplus(A^\oplus) &= \mu^\circ(A'), \text{ and} \\ \mu^\oplus(B^\oplus) &= \mu^\circ(B').\end{aligned}$$

Since $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$, thus $\mu^\circ(A' \cup B') = \mu^\circ(A') + \mu^\circ(B')$, so $\mu^\oplus(A^\oplus \cup B^\oplus) = \mu(A^\oplus) + \mu(B^\oplus)$, as desired. \square

◦ STEP 4: Conclude.

Let $A, B \in \mathcal{A}$. We wish to show $A \succsim B$ if and only if $\mu(A) \geq \mu(B)$. The idea of the proof is to construct, by algorithm, a list of pairs $(A, B) = (A_1, B_1), (A_2, B_2), (A_3, B_3), \dots$ such that for each i ,

- (i) $A_i \succsim B_i$ if and only if $A_{i+1} \succsim B_{i+1}$, and
- (ii) $\mu(A_i) \geq \mu(B_i)$ if and only if $\mu(A_{i+1}) \geq \mu(B_{i+1})$.

Moreover, the process is guaranteed to terminate with some pair (A_Ω, B_Ω) for which the statement “ $A_\Omega \succsim B_\Omega$ if and only if $\mu(A_\Omega) \geq \mu(B_\Omega)$ ” can be established, immediately proving “ $A \succsim B$ if and only if $\mu(A) \geq \mu(B)$,” as desired. Visually, it can be helpful to think of this list of pairs as a “logical ladder,” where the rails and rungs are formed of if-and-only-if arrows. For convenience, given $A_i, B_i, A_{i+1}, B_{i+1}$, let us say “we can continue with A_{i+1} and B_{i+1} ” as shorthand for statements (i) and (ii) above.

Given $A_i, B_i \in \mathcal{A}$, define the following four events:

$$\begin{aligned}A_i^\circ &\equiv A_i \cap S^\circ, \\ A_i^\oplus &\equiv A_i \cap S^\oplus, \\ B_i^\circ &\equiv B_i \cap S^\circ, \text{ and} \\ B_i^\oplus &\equiv B_i \cap S^\oplus.\end{aligned}$$

Moreover, define $n_i^\circ, n_i^\oplus \in \{0, 1, 2\}$ by:

$$\begin{aligned}n_i^\circ &\equiv |\{C \in \{A_i^\circ, B_i^\circ\} | C \neq \emptyset\}|, \text{ and} \\ n_i^\oplus &\equiv |\{C \in \{A_i^\oplus, B_i^\oplus\} | C \neq \emptyset\}|.\end{aligned}$$

Given A_i, B_i , the procedure is defined by following the earliest instruction that applies:

1. If $A_i = \emptyset$ or $B_i = \emptyset$, then terminate.

In the former case, $\emptyset \succsim B_i$ implies (by *monotonicity* and the [Domination Lemma](#)) $\emptyset \succsim B_i^\oplus$ and $\emptyset \succsim B_i^\circ$; thus $0 = \mu^\oplus(\emptyset) \geq \mu^\oplus(B_i^\oplus)$ and $0 = \mu^\circ(\emptyset) \geq \mu^\circ(B_i^\circ)$, so by construction $0 = \mu(\emptyset) = \mu(B_i)$. Conversely, $\mu(\emptyset) \geq \mu(B_i)$ implies $0 \geq \mu^\oplus(B_i^\oplus)$ and $0 \geq \mu^\circ(B_i^\circ)$; thus $\emptyset \succsim B_i^\oplus$ and $\emptyset \succsim B_i^\circ$, so by the [Domination Lemma](#) $\emptyset \succsim B_i$.

In the latter case, we have directly that $A_i \succsim \emptyset$ and $\mu(A_i) \geq \mu(\emptyset)$.

2. If $n_i^\circ = 0$ or $n_i^\oplus = 0$, then terminate; by construction of μ , $A_i \succsim B_i$ if and only if $\mu(A_i) \geq \mu(B_i)$.

3. If $n_i^\circ = 2$, then there are two cases.

- If $A_i^\circ \succsim B_i^\circ$, then by the [Carving Lemma](#), there is $C^\circ \subseteq A_i^\circ$ such that $C^\circ \sim B_i^\circ$. Define $B'_i \equiv (B_i \setminus B_i^\circ) \cup C^\circ$. By *separability* and additivity of μ , we can continue with A_i and B'_i . Define $A_{i+1} \equiv A_i \setminus C^\circ$ and $B_{i+1} \equiv B'_i \setminus C^\circ$. By *separability* and additivity of μ , we can continue with A_{i+1} and B_{i+1} .

- If $B_i^\circ \succsim A_i^\circ$, perform the operation analogous to that in the previous case.

In both cases, $n_{i+1}^\circ < n_i^\circ$ and $n_{i+1}^\oplus = n_i^\oplus$.

4. If $n_i^\circ = 1$ and $n_i^\oplus = 2$, then there are two cases.

- If $A_i^\circ = \emptyset$ and $A_i^\oplus \neq \emptyset$, then since by *monotonicity* $S^\circ \succsim S^\oplus \succsim A_i^\oplus$, by the [Carving Lemma](#) there is $A^\circ \in \mathcal{A}^\circ$ such that $A^\circ \sim A_i^\oplus$. Define $A_{i+1} \equiv A^\circ$ and define $B_{i+1} \equiv B_i$. By construction of μ , we can continue with A_{i+1} and B_{i+1} .

- If $B_i^\circ = \emptyset$ and $B_i^\oplus \neq \emptyset$, perform the operation analogous to that in the previous case.

In both cases, $n_i^\circ = 2$ and $n_{i+1}^\oplus = 1$.

5. If $n_i^\circ = 1$ and $n_i^\oplus = 1$, and neither A_i nor B_i is \emptyset , then there are two cases:

- If $A_i^\circ = \emptyset$ and $A_i^\oplus \neq \emptyset$, perform the operation analogous to that in the first case of the previous instruction.

- If $B_i^\circ = \emptyset$ and $B_i^\oplus \neq \emptyset$, perform the operation analogous to that in the second case of the previous instruction.

In both cases, $n_i^\circ = 2$ and $n_{i+1}^\oplus = 0$.

It is straightforward to verify that the process must terminate. By [Theorem V1](#), since $\mu \in \mathbb{M}(\mathcal{A})$ represents \succsim , thus $\mu \in \mathbb{M}^\sigma(\mathcal{A})$. It is clear from its construction that μ is the unique such measure. ■

Appendix 8

In this appendix, we prove the [Finite-Twin Lemma](#) and [Theorem 4](#).

- **FINITE-TWIN LEMMA:** If $(S, \mathcal{A}, \succsim)$ is a σ -catalogued qualitative probability space satisfying (*monotone*) *continuity* and 1^+ -*AS*, then for each $s \in S^\oplus$ such that $S^\circ \succsim \{s + 1, s + 2, \dots\}$, there is $T_s \in \mathcal{A}$ such that

- (i) $T_s \subseteq \{s + 1, s + 2, \dots\} \cup S^\circ$,

- (ii) $T_s \cap S^\oplus$ is finite, and
- (iii) $T_s \sim \{s\}$.

PROOF: Let $(S, \mathcal{A}, \succsim)$ and $s \in S^\oplus$ satisfy the hypotheses. By 1^+ -AS, there is pairwise-disjoint $\mathcal{B} \subseteq (\downarrow \{s\})$ such that $(\cup \mathcal{B}) \succ \{s\}$. For each $B \in \mathcal{B}$ and each $s' \in B \cap S^\oplus$, $s' > s$; else since $B \in (\downarrow \{s\})$, by *monotonicity* $\{s\} \succ B \succsim \{s'\} \succsim \{s\}$, contradicting $\{s\} \sim \{s\}$. Thus $(\cup \mathcal{B}) \subseteq \{s+1, s+2, \dots\} \cup S^\circ$.

We claim there is $A \subseteq (\cup \mathcal{B})$ such that $A \cap S^\oplus$ is finite and $A \succ \{s\}$. Indeed, for each $i \in \mathbb{N}$, define $A_i \equiv (\cup \mathcal{B}) \cap \{1, 2, \dots, i\}$. We cannot have that for each $i \in \mathbb{N}$, $\{s\} \succ A_i$, else by *continuity*, $\{s\} \succsim \lim A_i = (\cup \mathcal{B})$, contradicting $(\cup \mathcal{B}) \succ \{s\}$. Thus we can define i^* as the least i such that $A_i \succsim \{s\}$, and we can define $A \equiv A_{i^*}$. If $A \sim \{s\}$, then define $T_s \equiv A$ and we are done, so assume $A \succ \{s\}$.

If $A \cap S^\oplus \succ \{s\}$, then by *monotonicity*, $S^\circ \succsim \{s+1, s+2, \dots\} \succsim A \cap S^\oplus \succsim \{s\}$, so by the [Carving Lemma](#) there is $T_s \subseteq S^\circ$ such that $T_s \sim \{s\}$ and we are done. If $\{s\} \succsim A \cap S^\oplus$, then we follow an argument analogous to that in the proof of the [Carving Lemma](#). We begin by defining $V^+, V^- \subseteq [0, 1]$ by:

$$V^+ \equiv \{v \in [0, 1] \mid$$

$$V^+ \equiv \{v \in [0, \mu^\circ(A \cap S^\circ)] \mid C \in \mathcal{A}^\circ \text{ and } \mu^\circ(C) = v \text{ implies } (A \cap S^\oplus) \cup C \succsim \{s\}\}, \text{ and}$$

$$V^- \equiv \{v \in [0, \mu^\circ(A \cap S^\circ)] \mid C \in \mathcal{A}^\circ \text{ and } \mu^\circ(C) = v \text{ implies } \{s\} \succsim (A \cap S^\oplus) \cup C\}.$$

Through the analogous argument, there is $C \in \mathcal{A}^\circ$ such that $(A \cap S^\oplus) \cup C \sim \{s\}$; define $T_s \equiv (A \cap S^\oplus) \cup C$ and we are done. ■

• **THEOREM 4:** If $(S, \mathcal{A}, \succsim)$ is a σ -catalogued qualitative probability space satisfying (*monotone*) *continuity* and 3^+ -AS, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

PROOF: Let $(S, \mathcal{A}, \succsim)$ be a catalogued qualitative probability space satisfying *monotone continuity* and 3^+ -AS. If $(S, \mathcal{A}, \succsim)$ is purely-catalogued, then we are done by [Theorem 1](#), and if $S^\circ \sim \emptyset$ the proof with [Theorem 1](#) is trivial, so assume $S^\circ \succ \emptyset$.

◦ STEP 1: Define \succsim° and μ° .

For each pair $A, B \in \mathcal{A}^\circ$, define $A \succsim^\circ B$ if and only if $A \succsim B$. Since $S^\circ \succ \emptyset$, thus $(S^\circ, \mathcal{A}^\circ, \succsim^\circ)$ is an atomless qualitative probability space satisfying *monotone continuity*, so by [Theorem V2](#), there is a unique $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$ representing \succsim° . Moreover, μ° is infinitely-divisible. □

◦ STEP 2: For each ‘middle’ $m \in S^\oplus$, define the space $(S^{\leftarrow m}, \mathcal{A}^{\leftarrow m}, \succsim^{\leftarrow m})$, its associated $S^{\leftarrow m \oplus}$ and $\mathcal{A}^{\leftarrow m \oplus}$, the space $(S^{m \rightarrow}, \mathcal{A}^{m \rightarrow}, \succsim^{m \rightarrow})$, and its associated $S^{m \rightarrow \oplus}$ and $\mathcal{A}^{m \rightarrow \oplus}$.

Let $m \in S^\oplus$.

Define $S^{\leftarrow m} \equiv \{1, 2, \dots, m\} \cup S^\circ$ and define $\mathcal{A}^{\leftarrow m} \equiv \mathcal{A} \cap 2^{S^{\leftarrow m}}$. For each pair $A, B \in \mathcal{A}^{\leftarrow m}$, define $A \succsim^{\leftarrow m} B$ if and only if $A \succsim B$. Define $S^{\leftarrow m \oplus} \equiv \{1, 2, \dots, m\}$ and define $\mathcal{A}^{\leftarrow m \oplus} \equiv 2^{S^{\leftarrow m \oplus}}$.

Define $S^{m \rightarrow} \equiv \{m, m+1, \dots\} \cup S^\circ$ and define $\mathcal{A}^{m \rightarrow} \equiv \mathcal{A} \cap 2^{S^{m \rightarrow}}$. For each pair $A, B \in \mathcal{A}^{m \rightarrow}$, define $A \succsim^{m \rightarrow} B$ if and only if $A \succsim B$. Define $S^{m \rightarrow \oplus} \equiv \{m, m+1, \dots\}$ and define $\mathcal{A}^{m \rightarrow \oplus} \equiv 2^{S^{m \rightarrow \oplus}}$.

◦ STEP 3: Identify $m^{\leftarrow*} \in S^{\oplus}$ such that for each $m \geq m^{\leftarrow*}$, $(S^{\leftarrow m}, \mathcal{A}^{\leftarrow m}, \succsim^{\leftarrow m})$ is a finitely-catalogued qualitative probability space satisfying *monotone continuity* and 3-AS.

It is straightforward to verify that for each $m \in S^{\oplus}$, $(S^{\leftarrow m}, \mathcal{A}^{\leftarrow m}, \succsim^{\leftarrow m})$ is a finitely-catalogued qualitative probability space satisfying *monotone continuity*.

We wish to identify $m^{\leftarrow*}$ and verify that it is as desired. To begin, for each $s \in S^{\oplus}$, let us say that $t \geq s$ is a *truncation point* of s if there are

- (i) pairwise-disjoint $\mathcal{B} \subseteq (\downarrow \{s\}) \subseteq \mathcal{A}^{\leftarrow t}$, and
- (ii) $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \subseteq \mathcal{B}$ partitioning \mathcal{B} ,

such that for each $i \in \{1, 2, 3\}$, $(\cup \mathcal{B}_i) \succsim^{\leftarrow t} \{s\}$. Informally, if t is a truncation point of s , then $\{s\}$ is third-order swarmed by less-likely non-null events even if we ‘truncate’ at t by removing the tail of cores with higher index.

First, for each $s \in S^{\oplus}$, we construct a particular truncation point $\tau(s)$. Indeed, let $s \in S^{\oplus}$. Since \succsim satisfies 3⁺-AS, there are

- (i) pairwise-disjoint $\mathcal{B} \subseteq (\downarrow \{s\})$, and
- (ii) $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \subseteq \mathcal{B}'$ partitioning \mathcal{B} ,

such that for each $i \in \{1, 2, 3\}$, $(\cup \mathcal{B}_i) \succ \{s\}$. Let $i \in \{1, 2, 3\}$. For each $j \in \mathbb{N}$, define $B_j \equiv (\cup \mathcal{B}_i) \cap S^{\leftarrow j}$. Then $\lim B_j = (\cup \mathcal{B}_i)$, and by *separability*, (B_j) is increasing in likelihood. We cannot have that for each $j \in \mathbb{N}$, $\{s\} \succ B_j$, else by *continuity*, $\{s\} \succ (\cup \mathcal{B}_i)$, contradicting $(\cup \mathcal{B}_i) \succ \{s\}$. Thus we can define $j^*(i)$ as the least j such that $B_j \succsim \{s\}$. Define

$$\tau(s) \equiv \max\{s, j^*(1), j^*(2), j^*(3)\},$$

define $\mathcal{B} \equiv \mathcal{B} \cap \mathcal{A}^{\leftarrow \tau(s)}$, and for each $i \in \{1, 2, 3\}$, define $\mathcal{B}_i \equiv \mathcal{B}_i \cap \mathcal{A}^{\leftarrow \tau(s)}$. It is straightforward to verify that these collections of events are as desired, so that $\tau(s) \geq s$ is indeed a truncation point of s .

Second, we claim there is $s^* \in S^{\oplus}$ such that for each $s \geq s^*$, s is one of its own truncation points. Indeed, since μ° is infinitely-divisible and represents \succsim° , there are $A_1^{\circ}, A_2^{\circ}, A_3^{\circ}$ partitioning S° such that $A_1^{\circ} \sim A_2^{\circ} \sim A_3^{\circ}$. Since $S^{\circ} \succ \emptyset$, thus by *separability*, $A_1^{\circ} \succ \emptyset$. It cannot be that for each $s \in S^{\oplus}$, $\{s\} \succ A_1^{\circ}$; else by *continuity*, $\emptyset = \lim \{s\} \succsim A_1^{\circ}$, contradicting $A_1^{\circ} \succ \emptyset$. Thus we can define s^* to be the least $s \in S^{\oplus}$ such that $A_1^{\circ} \succsim \{s\}$. To see s^* is as desired, let $s \geq s^*$. Then $A_1^{\circ} \succsim \{s^*\} \succsim \{s\}$, so by the [Carving Lemma](#), there is $A_s^{\circ} \subseteq A_1^{\circ}$ such that $A_s^{\circ} \sim \{s\} \succ \emptyset$. Then $\mu^{\circ}(A_s^{\circ}) > 0$. Since μ° is infinitely-divisible, for each $i \in \{1, 2, 3\}$, there is \mathcal{B}_i be partitioning A_i° such that for each $B \in \mathcal{B}_i$, $\mu^{\circ}(A_s^{\circ}) > \mu^{\circ}(B) > 0$. Define $\mathcal{B} = \cup \mathcal{B}_i$. It is straightforward to verify that these collections of events are as desired, so that s is indeed a truncation point of s .

Finally, define $m^{\leftarrow*} \equiv \max\{\tau(1), \tau(2), \dots, \tau(s^* - 1), s^*\}$. To see that $m^{\leftarrow*}$ is as desired, let $m \geq m^{\leftarrow*}$. Then the cores in $(S^{\leftarrow m}, \mathcal{A}^{\leftarrow m}, \succsim^{\leftarrow m})$ are $\{1, 2, \dots, m\}$. By construction, for each $s \in \{1, 2, \dots, m\}$, there is a truncation point of s in $\{1, 2, \dots, m\}$. Thus $(S^{\leftarrow m}, \mathcal{A}^{\leftarrow m}, \succsim^{\leftarrow m})$ satisfies 3-AS, as desired. \square

◦ STEP 4: Identify $m^{*\rightarrow} \in S^{\oplus}$ such that for each $m \geq m^{*\rightarrow}$, $(S^{m\rightarrow}, \mathcal{A}^{m\rightarrow}, \succsim^{m\rightarrow})$ is a catalogued probability space satisfying *monotone continuity* and *unlikely cores*.

It is straightforward to verify that for each $m \in S^{\oplus}$, $(S^{m\rightarrow}, \mathcal{A}^{m\rightarrow}, \succsim^{m\rightarrow})$ is a catalogued qualitative probability space satisfying *monotone continuity*.

We wish to identify $m^{*\rightarrow}$ and verify that it is as desired. It cannot be that for each $s \in S^\oplus$, $\{s, s+1, \dots\} \succ S^\circ$; else by *continuity*, $\emptyset = \lim_{s \rightarrow \infty} \{s, s+1, \dots\} \lesssim S^\circ$, contradicting $S^\circ \succ \emptyset$. Let $m^{*\rightarrow}$ be the least $s \in S^\oplus$ such that $S^\circ \lesssim \{s, s+1, \dots\}$. To see that $m^{*\rightarrow}$ is as desired, let $m \geq m^{*\rightarrow}$. Then the cores in $(S^{m\rightarrow}, \mathcal{A}^{m\rightarrow}, \lesssim^{m\rightarrow})$ are $\{m, m+1, \dots\}$. By *monotonicity*, $S^\circ \lesssim \{m^{*\rightarrow}, m^{*\rightarrow}+1, \dots\} \lesssim \{m, m+1, \dots\}$; thus $(S^{m\rightarrow}, \mathcal{A}^{m\rightarrow}, \lesssim^{m\rightarrow})$ satisfies *unlikely cores*, as desired. \square

◦ STEP 5: Define m^* such that for each $m \geq m^*$, there is a unique $\mu^m \in \mathbb{M}^\sigma(\mathcal{A})$ such that

- (i) for each pair $A, B \in \mathcal{A}^{\leftarrow m}$, $A \lesssim B$ if and only if $\mu^m(A) \geq \mu^m(B)$, and
- (ii) for each pair $A, B \in \mathcal{A}^{m+1\rightarrow}$, $A \lesssim B$ if and only if $\mu^m(A) \geq \mu^m(B)$.

Define $m^* \equiv \max\{m^{\leftarrow*}, m^{*\rightarrow}\}$. Let $m \geq m^*$. To define $\mu^m \in \mathbb{M}^\sigma(\mathcal{A})$, we first define $\mu^{\leftarrow m} \in \mathbb{M}^\sigma(\mathcal{A}^{\leftarrow m})$ and $\mu^{m+1\rightarrow} \in \mathbb{M}^\sigma(\mathcal{A}^{m+1\rightarrow})$.

Since $m \geq m^* \geq m^{\leftarrow*}$, by Step 3, $(S^{\leftarrow m}, \mathcal{A}^{\leftarrow m}, \lesssim^{\leftarrow m})$ is a finitely-catalogued qualitative probability space satisfying *monotone continuity* and 3-AS. Thus by Theorem 2, there is a unique $\mu^{\leftarrow m} \in \mathbb{M}^\sigma(\mathcal{A}^{\leftarrow m})$ representing $\lesssim^{\leftarrow m}$.

Since $m \geq m^* \geq m^{*\rightarrow}$, $(S^{m+1\rightarrow}, \mathcal{A}^{m+1\rightarrow}, \lesssim^{m+1\rightarrow})$ is a finitely-catalogued qualitative probability space satisfying *monotone continuity* and *unlikely cores*. Thus by Theorem 3, there is a unique $\mu^{m+1\rightarrow} \in \mathbb{M}^\sigma(\mathcal{A}^{m+1\rightarrow})$ representing $\lesssim^{m+1\rightarrow}$.

Assume we have $\mu^m \in \mathbb{M}^\sigma(\mathcal{A})$ satisfying the requirements. Then for each $A \in \mathcal{A}^{\leftarrow m}$, we can define $\mu^{\leftarrow}(A)$ by:

$$\mu^{\leftarrow}(A) \equiv \frac{\mu^m(A)}{\mu^m(S^{\leftarrow m})}.$$

It is straightforward to verify that $\mu^{\leftarrow} \in \mathbb{M}^\sigma(\mathcal{A}^{\leftarrow m})$ and μ^{\leftarrow} represents $(S^{\leftarrow m}, \mathcal{A}^{\leftarrow m}, \lesssim^{\leftarrow m})$; thus by the uniqueness of $\mu^{\leftarrow m}$, $\mu^m(A) = \mu^{\leftarrow m}(A)\mu^m(S^{\leftarrow m})$. Since μ^m is additive, $\mu^m(A) = \mu^{\leftarrow m}(A)[\mu^m(S^\circ) + \mu^m(S^{\leftarrow m\oplus})]$.

Similarly, for each $A \in \mathcal{A}^{m+1\rightarrow}$, $\mu^m(A) = \mu^{m+1\rightarrow}(A)[\mu^m(S^\circ) + \mu^m(S^{m+1\rightarrow\oplus})]$.

Altogether, for each $A \in (\mathcal{A}^{\leftarrow m} \cup \mathcal{A}^{m+1\rightarrow})$, $\mu^m(A)$ can be written in terms of $\mu^m(S^\circ)$, $\mu^m(S^{\leftarrow m\oplus})$, and $\mu^m(S^{m+1\rightarrow\oplus})$. For clarity, let us define the ‘unknown’ $x_1, x_2, x_3 > 0$ and the ‘known’ $a, b, c, d > 0$ by:

$$\begin{aligned} x_1 &\equiv \mu^m(S^\circ), \\ x_2 &\equiv \mu^m(S^{\leftarrow m\oplus}), \\ x_3 &\equiv \mu^m(S^{m+1\rightarrow\oplus}), \\ a &\equiv \mu^{\leftarrow m}(S^\circ), \\ b &\equiv \mu^{m+1\rightarrow}(S^\circ), \\ c &\equiv \mu^{\leftarrow m}(S^{\leftarrow m\oplus}), \text{ and} \\ d &\equiv \mu^{m+1\rightarrow}(S^{m+1\rightarrow\oplus}). \end{aligned}$$

Then we have the following system of equations:

$$\begin{aligned} x_1 &= a(x_1 + x_2), \\ x_1 &= b(x_1 + x_3), \\ x_2 &= c(x_1 + x_2), \\ x_3 &= d(x_1 + x_3), \text{ and} \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

As each variable is positive, it is straightforward that this system has a unique solution.

To define μ , first define $\mu^m(S^\circ)$, $\mu^m(S^{\leftarrow m \oplus})$, and $\mu^m(S^{m+1 \rightarrow \oplus})$ using the solution to the above system. Then, for each $A \in (\mathcal{A}^{\leftarrow m} \cup \mathcal{A}^{m+1 \rightarrow})$, define $\mu^m(A)$ using $\mu^m(S^\circ)$, $\mu^m(S^{\leftarrow m \oplus})$, and $\mu^m(S^{m+1 \rightarrow \oplus})$ as specified above. Finally, for each $A \in \mathcal{A}$, define

$$\mu^m(A) \equiv \mu^m(A \cap S^\circ) + \mu^m(A \cap S^{\leftarrow m \oplus}) + \mu^m(A \cap S^{m+1 \rightarrow \oplus}).$$

By construction, $\mu^m(S) = 1$. Moreover, the restriction of μ^m to $\mathcal{A}^{\leftarrow m}$ is a positive multiple of $\mu^{\leftarrow m}$, and the restriction of μ^m to $\mathcal{A}^{m+1 \rightarrow}$ is a positive multiple of $\mu^{m+1 \rightarrow}$. Since $\mu^{\leftarrow m}$ is σ -additive on $\mathcal{A}^{\leftarrow m}$ and $\mu^{m+1 \rightarrow}$ is σ -additive on $\mathcal{A}^{m+1 \rightarrow}$, thus by construction, μ^m is σ -additive across \mathcal{A} . Since $\mu^{\leftarrow m}$ represents $\succsim^{\leftarrow m}$ and $\mu^{m+1 \rightarrow}$ represents $\succsim^{m+1 \rightarrow}$, altogether μ^m is as desired. It is clear from its construction that μ^m is the unique such measure. \square

◦ STEP 6: For each pair $m, m' \geq m^*$, $\mu^m = \mu^{m'}$; define μ to be this measure.

We prove that for each $m \geq m^*$, $\mu^m = \mu^{m+1}$. Let $m \geq m^*$. By Step 5, to show $\mu^m = \mu^{m+1}$, it suffices to show

(i) for each pair $A, B \in \mathcal{A}^{\leftarrow m}$, $A \succsim B$ if and only if $\mu^{m+1}(A) \geq \mu^{m+1}(B)$, and

(ii) for each pair $A, B \in \mathcal{A}^{m+1 \rightarrow}$, $A \succsim B$ if and only if $\mu^{m+1}(A) \geq \mu^{m+1}(B)$,

since μ^m is the unique measure satisfying (i) and (ii). Moreover, (i) is immediate from the construction of μ^{m+1} , as $\mathcal{A}^{\leftarrow m} \subseteq \mathcal{A}^{\leftarrow m+1}$. Thus it remains to show (ii).

By Step 4 and Step 5, $S^\circ \succsim \{m+1\}$. Thus by the [Carving Lemma](#), there is $A_{m+1}^\circ \in \mathcal{A}^\circ$ such that $A_{m+1}^\circ \sim \{m+1\}$. Since $A_{m+1}^\circ, \{m+1\} \in \mathcal{A}^{m+1 \rightarrow}$, thus $\mu^m(A_{m+1}^\circ) = \mu^m(\{m+1\})$. Since $A_{m+1}^\circ, \{m+1\} \in \mathcal{A}^{\leftarrow m+1}$, thus $\mu^{m+1}(A_{m+1}^\circ) = \mu^{m+1}(\{m+1\})$.

For each $A \in \mathcal{A}^{m+1 \rightarrow}$, define $v'(A), v''(A)$ by:

$$v'(A) \equiv \frac{\mu^m(A)}{\mu^m(S^{m+2 \rightarrow})}, \text{ and}$$

$$v''(A) \equiv \frac{\mu^{m+1}(A)}{\mu^{m+1}(S^{m+2 \rightarrow})}.$$

Let μ' be the restriction of v' to $\mathcal{A}^{m+2 \rightarrow}$, and similarly let μ'' be the restriction of v'' to $\mathcal{A}^{m+2 \rightarrow}$. Clearly, both μ' and μ'' are σ -measures representing $(S^{m+2 \rightarrow}, \mathcal{A}^{m+2 \rightarrow}, \succsim^{m+2 \rightarrow})$, so by the uniqueness of $\mu^{m+2 \rightarrow}$, $\mu' = \mu''$. Thus for each $A \in \mathcal{A}^{m+2 \rightarrow}$, $v'(A) = v''(A)$.

Since $\mu^m(A_{m+1}^\circ) = \mu^m(\{m+1\})$ and $\mu^{m+1}(A_{m+1}^\circ) = \mu^{m+1}(\{m+1\})$, thus $v'(A_{m+1}^\circ) = v'(\{m+1\})$ and $v''(A_{m+1}^\circ) = v''(\{m+1\})$, so

$$\begin{aligned} v'(\{m+1\}) &= v'(A_{m+1}^\circ) \\ &= v''(A_{m+1}^\circ) \\ &= v''(\{m+1\}). \end{aligned}$$

Since v' and v'' are additive, thus $v' = v''$. Since the restriction of μ^m to $\mathcal{A}^{m+1 \rightarrow}$ represents $(S^{m+1 \rightarrow}, \mathcal{A}^{m+1 \rightarrow}, \succsim^{m+1 \rightarrow})$, and since v' is the product of this restriction with a positive number, altogether $v' = v''$ represents $(S^{m+1 \rightarrow}, \mathcal{A}^{m+1 \rightarrow}, \succsim^{m+1 \rightarrow})$. Since the restriction of μ^{m+1} to $\mathcal{A}^{m+1 \rightarrow}$ is the product of v'' with a positive number, altogether we are done. \square

◦ STEP 7: Define S^{\leftarrow} and S^{\rightarrow} such that $S^\circ \succsim S^{\rightarrow}$, then show that for the partition $\mathcal{S} \equiv \{S^\circ, S^{\leftarrow}, S^{\rightarrow}\}$, for each pair $S^*, S^{**} \in \mathcal{S}$ and each pair $A, B \subseteq (S^* \cup S^{**})$, $A \succsim B$ if and only if $\mu(A) \geq \mu(B)$.

Define $S^{\leftarrow} \equiv S^{\leftarrow m^* \oplus}$, define $S^{\rightarrow} \equiv S^{m^*+1 \rightarrow \oplus}$, and define $\mathcal{S} \equiv \{S^\circ, S^{\leftarrow}, S^{\rightarrow}\}$. By Step 4 and Step 5, $S^\circ \succsim S^{\rightarrow}$. Clearly, \mathcal{S} is a partition. Let $S^*, S^{**} \in \mathcal{S}$ and let $A, B \in \mathcal{A}$ with $A, B \subseteq (S^* \cup S^{**})$.

CASE 1: $S^*, S^{**} \in \{S^\circ, S^{\leftarrow}\}$. Then we are done by Step 5 and Step 6.

CASE 2: $S^*, S^{**} \in \{S^\circ, S^{\rightarrow}\}$. Then we are done by Step 5 and Step 6.

CASE 3: $\{S^*, S^{**}\} = \{S^{\leftarrow}, S^{\rightarrow}\}$. Let $C \in \mathcal{A}$ with $C \subseteq S^{\leftarrow} \cup S^{\rightarrow}$, define $C^{\leftarrow} \equiv C \cap S^{\leftarrow}$, and define $C^{\rightarrow} \equiv C \cap S^{\rightarrow}$. By *monotonicity*, $S^\circ \succsim S^{\rightarrow} \succsim C^{\rightarrow}$, so by the [Carving Lemma](#), there is $C^\circ \in \mathcal{A}^\circ$ such that $C^\circ \sim C^{\rightarrow}$. By Case 2, $\mu(C^\circ) = \mu(C^{\rightarrow})$. Define $C' \equiv C^{\leftarrow} \cup C^\circ$. By *separability*, $C' \sim C$, and by additivity of μ , $\mu(C') = \mu(C)$. Since C was arbitrary, thus there are $A', B' \in \mathcal{A}$ with $A', B' \subseteq (S^{\leftarrow} \cup S^\circ)$ such that $A \sim A'$, $B \sim B'$, $\mu(A) = \mu(A')$, and $\mu(B) = \mu(B')$. By Case 1, $A' \succsim B'$ if and only if $\mu(A') \geq \mu(B')$; thus $A \succsim B$ if and only if $\mu(A) \geq \mu(B)$, as desired. \square

◦ STEP 8: Conclude.

A simpler version of the argument in this step is found in Step 4 of the proof of [Theorem 3](#).

Let $A, B \in \mathcal{A}$. We wish to show $A \succsim B$ if and only if $\mu(A) \geq \mu(B)$. The idea of the proof is to construct, by algorithm, a list of pairs $(A, B) = (A_1, B_1), (A_2, B_2), (A_3, B_3), \dots$ such that for each i ,

- (i) $A_i \succsim B_i$ if and only if $A_{i+1} \succsim B_{i+1}$, and
- (ii) $\mu(A_i) \geq \mu(B_i)$ if and only if $\mu(A_{i+1}) \geq \mu(B_{i+1})$.

Moreover, the process is guaranteed to terminate with some pair (A_Ω, B_Ω) for which the statement “ $A_\Omega \succsim B_\Omega$ if and only if $\mu(A_\Omega) \geq \mu(B_\Omega)$ ” can be established, immediately proving “ $A \succsim B$ if and only if $\mu(A) \geq \mu(B)$,” as desired. Visually, it can be helpful to think of this list of pairs as a “logical ladder,” where the rails and rungs are formed of if-and-only-if arrows. For convenience, given $A_i, B_i, A_{i+1}, B_{i+1}$, let us say “we can continue with A_{i+1} and B_{i+1} ” as shorthand for statements (i) and (ii) above.

Given $A_i, B_i \in \mathcal{A}$, define the following six events:

$$\begin{aligned} A_i^\circ &\equiv A_i \cap S^\circ, \\ A_i^{\leftarrow} &\equiv A_i \cap S^{\leftarrow}, \\ A_i^{\rightarrow} &\equiv A_i \cap S^{\rightarrow}, \\ B_i^\circ &\equiv B_i \cap S^\circ, \\ B_i^{\leftarrow} &\equiv B_i \cap S^{\leftarrow}, \text{ and} \\ B_i^{\rightarrow} &\equiv B_i \cap S^{\rightarrow}. \end{aligned}$$

Moreover, define $n_i^\circ, n_i^{\leftarrow}, n_i^{\rightarrow} \in \{0, 1, 2\}$ by:

$$\begin{aligned} n_i^\circ &\equiv |\{C \in \{A_i^\circ, B_i^\circ\} | C \neq \emptyset\}|, \\ n_i^{\leftarrow} &\equiv |\{C \in \{A_i^{\leftarrow}, B_i^{\leftarrow}\} | C \neq \emptyset\}|, \text{ and} \\ n_i^{\rightarrow} &\equiv |\{C \in \{A_i^{\rightarrow}, B_i^{\rightarrow}\} | C \neq \emptyset\}|. \end{aligned}$$

Given A_i, B_i , the procedure is defined by following the earliest instruction that applies:

1. If $A_i = \emptyset$ or $B_i = \emptyset$, then terminate.

In the former case, $\emptyset \succsim B_i$ implies (by *monotonicity* and the [Domination Lemma](#)) $\emptyset \succsim B_i^{\leftarrow}$, $\emptyset \succsim B_i^{\rightarrow}$, and $\emptyset \succsim B_i^{\circ}$; thus by Step 5 and Step 6, $0 = \mu(\emptyset) \geq \mu(B_i^{\leftarrow})$, $0 = \mu(\emptyset) \geq \mu(B_i^{\rightarrow})$, and $0 = \mu(\emptyset) \geq \mu(B_i^{\circ})$; so by additivity $0 = \mu(\emptyset) = \mu(B_i)$. Conversely, $\mu(\emptyset) \geq \mu(B_i)$ implies $0 \geq \mu(B_i^{\leftarrow})$, $0 \geq \mu(B_i^{\rightarrow})$, and $0 \geq \mu(B_i^{\circ})$; thus by Step 5 and Step 6, $\emptyset \succsim B_i^{\leftarrow}$, $\emptyset \succsim B_i^{\rightarrow}$, and $\emptyset \succsim B_i^{\circ}$; so by the [Domination Lemma](#) $\emptyset \succsim B_i$.

In the latter case, we have directly that $A_i \succsim \emptyset$ and $\mu(A_i) \geq \mu(\emptyset)$.

2. If $n_i^{\circ} = 0$, or $n_i^{\leftarrow} = 0$, or $n_i^{\rightarrow} = 0$, then terminate; by Step 7, $A_i \succsim B_i$ if and only if $\mu(A_i) \geq \mu(B_i)$.

3. If $n_i^{\circ} = 2$, then there are two cases.

- If $A_i^{\circ} \succsim B_i^{\circ}$, then by the [Carving Lemma](#), there is $C^{\circ} \subseteq A_i^{\circ}$ such that $C^{\circ} \sim B_i^{\circ}$. Define $B_i' \equiv (B_i \setminus B_i^{\circ}) \cup C^{\circ}$. By *separability* and additivity of μ , we can continue with A_i and B_i' . Define $A_{i+1} \equiv A_i \setminus C^{\circ}$ and $B_{i+1} \equiv B_i' \setminus C^{\circ}$. By *separability* and additivity of μ , we can continue with A_{i+1} and B_{i+1} .

- If $B_i^{\circ} \succsim A_i^{\circ}$, perform the operation analogous to that in the previous case.

In both cases, $n_{i+1}^{\circ} < n_i^{\circ}$ and $n_{i+1}^{\rightarrow} = n_i^{\rightarrow}$.

4. If $n_i^{\circ} = 1$ and $n_i^{\rightarrow} = 2$, then there are two cases.

- If $A_i^{\circ} = \emptyset$ and $A_i^{\rightarrow} \neq \emptyset$, then since by *monotonicity* $S^{\circ} \succsim S^{\rightarrow} \succsim A_i^{\rightarrow}$, by the [Carving Lemma](#) there is $A^{\circ} \in \mathcal{A}^{\circ}$ such that $A^{\circ} \sim A_i^{\rightarrow}$. Define $A_{i+1} \equiv (A_i \setminus A_i^{\rightarrow}) \cup A^{\circ}$ and define $B_{i+1} \equiv B_i$. By *separability* and additivity of μ , we can continue with A_{i+1} and B_{i+1} .

- If $B_i^{\circ} = \emptyset$ and $B_i^{\rightarrow} \neq \emptyset$, perform the operation analogous to that in the previous case.

In both cases, $n_{i+1}^{\rightarrow} = 1$.

5. If $n_i^{\circ} = 1$, $n_i^{\rightarrow} = 1$, and neither A nor B is contained in S^{\leftarrow} , then there are two cases:

- If $A_i^{\circ} = \emptyset$ and $A_i^{\rightarrow} \neq \emptyset$, perform the operation analogous to that in the first case of the previous instruction.

- If $B_i^{\circ} = \emptyset$ and $B_i^{\rightarrow} \neq \emptyset$, perform the operation analogous to that in the second case of the previous instruction.

In both cases, $n_{i+1}^{\rightarrow} = 0$.

6. If $n_i^{\circ} = 1$, $n_i^{\rightarrow} = 1$, either A or B is contained in S^{\leftarrow} , and neither A_i nor B_i is \emptyset , then there are two cases:

- If $A_i = A_i^{\leftarrow}$, then define $s \equiv \max A_i$. By the [Finite-Twin Lemma](#), there is finite $T_s \subseteq \{s+1, s+2, \dots\} \cup S^{\circ}$ such that $T_s \sim \{s\}$. Define $m^{**} \equiv \max\{\max T_s, m^*\}$. Since $T_s, \{s\} \in \mathcal{A}^{\leftarrow m^{**}}$, thus by Step 5 and Step 6, $\mu(T_s) = \mu(\{s\})$. Define $A_{i+1} \equiv (A_i \setminus \{s\}) \cup T_s$ and define $B_{i+1} \equiv B_i$. By *separability* and additivity of μ , we can continue with A_{i+1} and B_{i+1} .

- If $B_i = B_i^{\leftarrow}$, perform the operation analogous to that in the previous case.

It is straightforward to verify that the process cannot indefinitely loop through the first four instructions. If the fifth instruction is ever reached, then the process terminates on the next step.

For each pair $A, B \subseteq S^\leftarrow$, let the statement A is *lexicographically greater than* B , written $A >_{LEX} B$, be defined by:

$$A >_{LEX} B \text{ if and only if } \min A \setminus B < \min B \setminus A.$$

Observe that each of the first five instructions sets $A_{i+1}^\leftarrow \equiv A_i^\leftarrow$ and $B_{i+1}^\leftarrow \equiv B_i^\leftarrow$. Moreover, each time the sixth instruction is reached, either $A_{i+1}^\leftarrow <_{LEX} A_i^\leftarrow$ or $B_{i+1}^\leftarrow <_{LEX} B_i^\leftarrow$. Because \mathcal{A}^\leftarrow is finite, there is some k for which $A_k^\leftarrow = \emptyset$ and there is some k' for which $B_{k'}^\leftarrow = \emptyset$. Define $\Omega = \max\{k, k'\}$, and the process terminates with A_Ω, B_Ω at the latest.

Altogether, $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ represents $(S, \mathcal{A}, \succsim)$. By Step 5 and Step 6, μ is the unique such measure. ■

Appendix 9

In this appendix, we prove [Theorem 5](#).

• **THEOREM 5:** A triple $(S, \mathcal{A}, \succsim)$ is a qualitative probability space satisfying (*monotone continuity* and 3^+ -*AS*) if and only if it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ such that for each $A \in \mathcal{A}$,

- (i) if A is an atom, $\mu(\cup(\downarrow A)) > 3\mu(A) > 0$, and
- (ii) if A contains no atoms, for each $\lambda \in [0, 1]$, there is $B \subseteq A$ such that $\mu(B) = \lambda\mu(A)$.

PROOF: It is trivial that our axioms are necessary for such a representation, so let $(S, \mathcal{A}, \succsim)$ be a qualitative probability space satisfying (*monotone continuity*) and 3^+ -*AS*.

◦ **STEP 1:** There is no increasing sequence of atoms.

Assume, by way of contradiction, there is $(A_i) \in (\mathcal{A}^\alpha)^\mathbb{N}$ such that $A_1 \prec A_2 \prec \dots$. Then for each pair $i, j \in \mathbb{N}$ with $i > j$, by *monotonicity*, $A_i \succ A_j \succsim A_i \cap A_j$, so since A_i is an atom, thus $A_i \cap A_j \sim \emptyset$. For each $i \in \mathbb{N}$, define $B_i \equiv A_i \setminus (\bigcup_{j < i} A_j) = A_i \setminus (\bigcup_{j < i} A_j \cap A_i)$; by *monotonicity* and *separability*, $(\bigcup_{j < i} A_i \cap A_j) \sim \emptyset$, so by *separability* $B_i \sim A_i$. By construction, the B_i are pairwise-disjoint. But then for each $i \in \mathbb{N}$, $B_i \sim A_i \succsim A_1$, so by *continuity*, $\emptyset = \lim B_i \succsim A_1$, contradicting that A_1 is an atom. □

◦ **STEP 2:** Construct index set I and pairwise-disjoint atoms $(A_i)_{i \in I}$, indexed in non-increasing order of likelihood, such that $S^\circ \equiv S \setminus (\cup A_i)$ contains no atoms.

If there are no atoms, define $I \equiv \emptyset$ and we are done; otherwise, we proceed inductively. For the base step, by Step 1, there is a \succsim -maximum A_1 in \mathcal{A}^α , and for each $B \in \mathcal{A}^\alpha$ such that $B \subseteq S \setminus A_1$, $A_1 \succsim B$. For the inductive step, assume that $k \in \mathbb{N}$ and we have pairwise-disjoint atoms A_1, A_2, \dots, A_k such that

- (1) $A_1 \succsim A_2 \succsim \dots \succsim A_k$, and
- (2) for each $B \in \mathcal{A}^\alpha$ such that $B \subseteq S \setminus \bigcup_{i=1}^k A_i$, $A_k \succsim B$.

Define $\mathcal{A}_k^\alpha \subseteq \mathcal{A}^\alpha$ by:

$$\mathcal{A}_k^\alpha \equiv \{B \in \mathcal{A}^\alpha \mid B \subseteq S \setminus \bigcup_{i=1}^k A_i\}.$$

If \mathcal{A}_k^α is empty, then we are done. If \mathcal{A}_k^α is nonempty, then by Step 1 it has \preceq -maximum A_{k+1} . Since $A_{k+1} \in \mathcal{A}^\alpha$ and $A_{k+1} \subseteq S \setminus \bigcup_{i=1}^k A_i$, thus by the inductive hypothesis, A_1, A_2, \dots, A_{k+1} are pairwise-disjoint atoms and $A_1 \preceq A_2 \preceq \dots \preceq A_{k+1}$. By construction, for each $B \in \mathcal{A}^\alpha$ such that $B \subseteq S \setminus \bigcup_{i=1}^{k+1} A_i$, $A_{k+1} \preceq B$.

Define $S^\circ \equiv S \setminus (\bigcup A_i)$. If construction terminates at some step, then $I = \{1, 2, \dots, |I|\}$; otherwise $I = \mathbb{N}$. If I is empty or finite, then S° contains no atoms by construction. If $I = \mathbb{N}$, then assume, by way of contradiction, there is an atom $A \subseteq S^\circ$. By construction, for each $i \in \mathbb{N}$, $A_i \preceq A$. But then by *continuity*, $\emptyset = \lim A_i \preceq A$, contradicting that A is an atom. \square

◦ STEP 3: Construct $(S^*, \mathcal{A}^*, \preceq^*)$ and the mappings Ψ, Ψ', Ψ^* .

Informally, to construct the catalogued space, we replace each A_i with $\{i\}$. Formally, define $S^* \equiv S^\circ \cup I$. Let \mathcal{A}^* be the σ -algebra generated by $(2^{S^\circ} \cap \mathcal{A}) \cup 2^I$; that is, $A \in \mathcal{A}^*$ if and only if there are $B_1 \subseteq S^\circ$ and $B_2 \subseteq I$ such that (i) $B_1 \in \mathcal{A}$, and (ii) $A = B_1 \cup B_2$.

Let $\mathcal{A}' \subseteq \mathcal{A}$ be the collection of events that contain any intersected A_i :

$$\mathcal{A}' \equiv \{A \in \mathcal{A} \mid \text{for each } i \in I, A \cap A_i \in \{\emptyset, A_i\}\}$$

Define $\Psi : \mathcal{A} \rightarrow \mathcal{A}'$, $\Psi' : \mathcal{A}' \rightarrow \mathcal{A}^*$, and $\Psi^* : \mathcal{A}^* \rightarrow \mathcal{A}'$ by:

$$\begin{aligned} \Psi(A) &\equiv (A \cap S^\circ) \cup \left(\bigcup \{A_i \in \mathcal{A} \mid A \cap A_i \sim A_i\} \right). \\ \Psi'(A) &\equiv (A \cap S^\circ) \cup \{i \in I \mid A \cap A_i = A_i\} \\ \Psi^*(A) &\equiv \Psi'^{-1}(A). \end{aligned}$$

Define \preceq^* by: for each pair $A, B \in \mathcal{A}^*$, $A \preceq^* B$ if and only if $\Psi^*(A) \preceq \Psi^*(B)$. \square

◦ STEP 4: Ψ preserves likelihood.

We wish to prove that for each $A \in \mathcal{A}$, $A \sim \Psi(A)$. Indeed, define $B_0 \equiv A \cap S^\circ$ and $C_0 \equiv \Psi(A) \cap S^\circ$, and define $(B_i), (C_i) \in \mathcal{A}^I$ recursively by:

$$\begin{aligned} B_{i+1} &\equiv B_i \cup (A \cap A_{i+1}) \\ C_{i+1} &\equiv C_i \cup (\Psi(A) \cap A_{i+1}). \end{aligned}$$

We claim that for each $i \in I$, $B_i \sim C_i$, which we prove by induction. For the base step, $B_0 = C_0$. For the inductive step, assume that $B_i \sim C_i$. By definition of Ψ , $(A \cap A_{i+1}) \sim (\Psi(A) \cap A_{i+1})$. By construction, B_i and $(A \cap A_{i+1})$ are disjoint, so by the [Domination Lemma](#), $B_{i+1} \preceq C_{i+1}$. By construction, C_i and $(\Psi(A) \cap A_{i+1})$ are disjoint, so by the [Domination Lemma](#), $C_{i+1} \preceq B_{i+1}$. Thus $B_{i+1} \sim C_{i+1}$ as desired.

If I is finite, then $A = B_{|I|}$ and $\Psi(A) = C_{|I|}$, so $A \sim \Psi(A)$ as desired. Otherwise, $A = \lim B_i$ and $C = \lim C_i$, so by the [Limit-Order Lemma](#) twice, $A \sim \Psi(A)$ as desired.

◦ STEP 5: Ψ, Ψ' , and Ψ^* each preserve disjoint pairs.

We wish to prove that for each of these mappings, the images of any disjoint pair in the domain form a disjoint pair in the co-domain. The trivial proofs for Ψ' and Ψ^* are omitted.

Let $B, C \in \mathcal{A}$ be disjoint. Since $(B \cap S^\circ) = (\Psi(B) \cap S^\circ)$ and $(C \cap S^\circ) = (\Psi(C) \cap S^\circ)$, thus $(\Psi(B) \cap S^\circ)$ and $(\Psi(C) \cap S^\circ)$ are disjoint.

We claim for each $i \in I$, $(\Psi(B) \cap A_i)$ and $(\Psi(C) \cap A_i)$ are disjoint. Indeed, let $i \in I$. Assume, by way of contradiction, $(B \cap A_i)$ and $(C \cap A_i)$ are both non-null. Then since A_i is an atom, $(B \cap A_i) \sim A_i$ and $(C \cap A_i) \sim A_i \succ \emptyset$. But then since $(B \cap A_i)$ and $(C \cap A_i)$ are disjoint, thus by the [Domination Lemma](#) $(B \cap A_i) \cup (C \cap A_i) \succ A_i \cup \emptyset = A_i$, contradicting *monotonicity*. Thus either $(B \cap A_i)$ or $(C \cap A_i)$ is null, so either $\Psi(B \cap A_i)$ or $\Psi(C \cap A_i)$ is empty, so $\Psi(B \cap A_i)$ and $\Psi(C \cap A_i)$ are disjoint, as desired.

Since $S = S^\circ \cup (\cup A_i)$, and since $\Psi(B)$ and $\Psi(C)$ do not intersect in S° or in any A_i , thus $\Psi(B)$ and $\Psi(C)$ are disjoint, as desired.

◦ STEP 6: Verify $(S^*, \mathcal{A}^*, \succsim^*)$ is a catalogued qualitative probability space satisfying *monotone continuity* and 3^+ -AS.

We verify each property in sequence:

ORDER: To see that \succsim^* is complete, let $A, B \in \mathcal{A}^*$. Since \succsim is complete, $\Psi^*(A) \succsim \Psi^*(B)$; thus $A \succsim^* B$ as desired.

To see that \succsim^* is transitive, let $A, B, C \in \mathcal{A}^*$ such that $A \succsim^* B$ and $B \succsim^* C$. Then $\Psi^*(A) \succsim \Psi^*(B)$ and $\Psi^*(B) \succsim \Psi^*(C)$, so by transitivity of \succsim , $\Psi^*(A) \succsim \Psi^*(C)$; thus $A \succsim^* C$ as desired.

SEPARABILITY: Let $A, B, C \in \mathcal{A}^*$ such that $A \cap C = B \cap C = \emptyset$. Since Ψ^* preserves disjoint pairs, $\Psi^*(A) \cap \Psi^*(C) = \Psi^*(B) \cap \Psi^*(C) = \emptyset$. Since \succsim is *separable*, $\Psi^*(A) \succsim \Psi^*(B)$ if and only if $\Psi^*(A) \cup \Psi^*(C) \succsim \Psi^*(B) \cup \Psi^*(C)$; thus $A \succsim^* B$ if and only if $A \cup C \succsim^* B \cup C$, as desired.

MONOTONICITY: Let $A, B \in \mathcal{A}$ with $A \subseteq B$. Then $\Psi^*(A) \subseteq \Psi^*(B)$. Since \succsim is *monotonic*, $\Psi^*(B) \succsim \Psi^*(A)$; thus $B \succsim^* A$, as desired.

NONDEGENERACY: Since \succsim is *nondegenerate*, there are $A, B \in \mathcal{A}$ such that $A \succ B$. Since Ψ preserves likelihood, $\Psi(A) \succ \Psi(B)$; thus $\Psi'(\Psi(A)) \succ^* \Psi'(\Psi(B))$, as desired.

CONTINUITY: Let $A \in \mathcal{A}^*$ and let $(B_i) \in (\mathcal{A}^*)^\mathbb{N}$ be convergent. Define $B \equiv \lim B_i$. Clearly, $(\Psi^*(B_i))$ is convergent with $\Psi^*(B) = \lim \Psi^*(B_i)$.

If for each $i \in \mathbb{N}$, $B_i \succsim^* A$, then for each $i \in \mathbb{N}$, $\Psi^*(B_i) \succsim \Psi^*(A)$. Since \succsim is *continuous*, $\Psi^*(B) \succsim \Psi^*(A)$; thus $B \succsim^* A$, as desired.

If for each $i \in \mathbb{N}$, $A \succsim^* B_i$, then by a similar argument $A \succsim^* B$, as desired.

3^+ -AS: Let $A \in \mathcal{A}^*$ be an atom. Then $\Psi^*(A) \in \mathcal{A}$ is an atom. Since \succsim satisfies 3-AS, there are

(i) pairwise-disjoint $\mathcal{B} \subseteq (\downarrow \Psi^*(A))$, and

(ii) $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \subseteq \mathcal{B}'$ partitioning \mathcal{B} ,

such that for each $i \in \{1, 2, 3\}$, $(\cup \mathcal{B}_i) \succ \Psi^*(A)$.

Define $\mathcal{B}' \subseteq \mathcal{A}'$ and $\mathcal{B}^* \subseteq \mathcal{A}^*$ by:

$$\begin{aligned}\mathcal{B}' &\equiv \{\Psi(B) | B \in \mathcal{B}\} \\ \mathcal{B}^* &\equiv \{\Psi'(B) | B \in \mathcal{B}'\}\end{aligned}$$

Similarly, for each $i \in \{1, 2, 3\}$, define $\mathcal{B}'_i \subseteq \mathcal{A}'$ and $\mathcal{B}^*_i \subseteq \mathcal{A}^*$ by:

$$\begin{aligned}\mathcal{B}'_i &\equiv \{\Psi(B) | B \in \mathcal{B}_i\} \\ \mathcal{B}^*_i &\equiv \{\Psi'(B) | B \in \mathcal{B}'_i\}\end{aligned}$$

Since Ψ preserves likelihood and disjoint pairs,

- (i) $\mathcal{B}' \subseteq (\downarrow \Psi^*(A))$,
- (ii) \mathcal{B}' is pairwise-disjoint, and
- (iii) $\mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3 \subseteq \mathcal{B}'$ partition \mathcal{B}' .

Moreover, let $i \in \{1, 2, 3\}$. We first claim \mathcal{B}_i is finite or countably infinite. Indeed, there is no non-decreasing sequence $(B_j) \in (\mathcal{B}_i)^\mathbb{N}$; else since \mathcal{B}_i is pairwise-disjoint, $\emptyset = \lim B_j$, so by *continuity* $\emptyset \succsim B_1$, contradicting $B_1 \in (\downarrow \Psi^*(A))$. If \mathcal{B}_i is not finite, then we can construct $(B_j) \in (\mathcal{B}'_i)^\mathbb{N}$ by iteratively taking the most-likely remaining event as in Step 2, and since \mathcal{B}_i is pairwise-disjoint, $\emptyset = \lim B_j$. In this case, $\{B_j\} = \mathcal{B}_i$; else there is $B \in \mathcal{B}_i \setminus \{B_j\}$, and by construction, for each $j \in \mathbb{N}$, $B_j \succsim B$, so by *continuity* $\emptyset = \lim B_j \succsim B$, contradicting $B \in (\downarrow \Psi^*(A))$. Thus \mathcal{B}_i is finite or countably infinite, as desired.

Label the members of \mathcal{B}_i by $\{B_1, B_2, \dots, B_j, \dots\} = \{B_j\}_{j \in J}$, where $J \subseteq \mathbb{N}$ may or may not be finite. For each $j \in J$, define $B'_j \equiv \Psi(B_j)$, define $C_j \equiv \cup_{k \leq j} B_k$, and define $C'_j \equiv \cup_{k \leq j} \Psi(B_k)$.

We claim that for each $j \in J$, $C_j \sim C'_j$, which we prove by induction through an argument similar to that in Step 4. For the base step, since Ψ preserves likelihood, $C_1 \sim C'_1$. For the inductive step, assume that $C_j \sim C'_j$. Since Ψ preserves likelihood, $B_{j+1} \sim B'_{j+1}$. By construction, C_j and B_{j+1} are disjoint, so by the [Domination Lemma](#), $C'_{j+1} \succsim C'_j$. By construction, C'_j and B'_{j+1} are disjoint, so by the [Domination Lemma](#), $C'_{j+1} \succsim C'_{j+1}$. Thus $C_{j+1} \sim C'_{j+1}$, as desired.

If J is finite, then $\cup \mathcal{B}_i = C_{|J|}$ and $\cup \mathcal{B}'_i = C'_{|J|}$, so $\cup \mathcal{B}_i \sim \cup \mathcal{B}'_i$. Otherwise, $\mathcal{B}_i = \lim C_j$ and $\mathcal{B}'_i = \lim C'_j$, so by the [Limit-Order Lemma](#) twice, $\cup \mathcal{B}_i \sim \cup \mathcal{B}'_i$. Thus $\cup \mathcal{B}'_i \succ \Psi^*(A)$.

By definition of Ψ' , since Ψ' preserves disjoint pairs, and since Ψ' is the inverse of Ψ^* , it follows from the above properties of $\mathcal{B}', \mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$ that:

- (i) $\mathcal{B}^* \subseteq (\downarrow A)$,
- (ii) \mathcal{B}^* is pairwise-disjoint,
- (iii) $\mathcal{B}^*_1, \mathcal{B}^*_2, \mathcal{B}^*_3 \subseteq \mathcal{B}^*$ partition \mathcal{B}^* , and
- (iv) for each $i \in \{1, 2, 3\}$, $\cup \mathcal{B}^*_i \succ^* A$,

as desired.

CATALOGUED: To see that $(S^*, \mathcal{A}^*, \succsim^*)$ is catalogued, simply take $S^\oplus = I$. \square

◦ STEP 7: $(S^*, \mathcal{A}^*, \succsim^*)$ has representation $\mu^* \in \mathbb{M}^\sigma(\mathcal{A}^*)$ with the desired properties.

Since $(S^*, \mathcal{A}^*, \succsim^*)$ is a catalogued qualitative probability space satisfying *continuity* and 3^+ -AS, this is immediate from [Theorem 2](#) and [Theorem 4](#).

◦ STEP 8: $(S, \mathcal{A}, \succsim)$ has unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ with the desired properties.

For each $A \in \mathcal{A}^*$, define $\mu(\Psi^*(A)) \equiv \mu^*(A)$. For each $i \in I$ and each $B \subseteq A_i$, define $\mu(B) \equiv \mu(A_i)$ if $B \sim A_i$ and define $\mu(B) \equiv 0$ otherwise. It is straightforward to complete the definition of μ to be a σ -measure and to verify its properties. ■

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